ELEMENTARY LESSONS

WITH

NUMERICAL EXAMPLES

IN

PRACTICAL MECHANICS

AND

MACHINE DESIGN

BY

ROBERT GORDON BLAINE M.E.

ASSOC. M. INST. C.E. &c.

SENIOR DEMONSTRATOR AND LECTURER IN THE MECHANICAL
ENGINEERING DEPARTMENT OF THE FINSBURY TECHNICAL COLLEGE
CITY AND GUILDS OF LONDON INSTITUTE

WITH AN

INTRODUCTION BY PROFESSOR JOHN PERRY, M.E., D.Sc., F.R.S.

THIRD EDITION REVISED

WITH 79 ILLUSTRATIONS

CASSELL AND COMPANY, LIMITED

LONDON, PARIS & MELBOURNE

1894

LA BELLE SAUVAGE

INTRODUCTION.

Our system of teaching Practical Mechanics at the Finsbury Technical College involves—first, two lectures; secondly, two hours' numerical exercise work to illustrate the lectures; and thirdly, three hours' mechanical laboratory work, per week. Mr. Blaine has, in this book, gathered together the best of the numerical examples set by him and worked out during the last few years by students in our exercise classes, and they may be taken as illustrating, not merely my lectures, but my book on "Practical Mechanics," and Prof. Unwin's book on "Machine Design."

It will be observed that they are all of a much more practical character, that is, they are more calculated to educate the practical mechanical engineer than the examples usually to be met with in books on Mechanics. The writers of ordinary books on Mechanics are unfortunately writing text-books or cram-books to be used in preparing for certain examinations; the curious numerical examples given have really very little to do with the subject of Mechanics, but they illustrate very well the occult phenomena described by their authors.

It is perfectly well known that the sort of knowledge which such authors try to give is treated with a certain amount of contempt by all practical engineers

Self-conceit and real ignorance are supposed to be the characteristics of their most energetic readers.

The teaching power of this collection of examples has already been well tried. Students who have gone through them show that they have obtained a real *working knowledge* of the application of the principles of Mechanics to Engineering and Machine Design, and that their knowledge is always ready for use. Each exercise fixes firmly in the mind of the student the fact that a certain principle is of importance outside examination rooms. He is not tempted to calculate in cases where actual experimental trial and observation will show him the best solution of a problem, and when he works out an answer which every practical illiterate mechanic knows to be ten times too great, he will not complacently rest satisfied with this absurd answer and talk about its being "theoretically" right.

JOHN PERRY.

10, *Penywern Road*,
4th March, 1888.

PREFACE TO SECOND EDITION.

THE book has been to a large extent re-written, and much additional matter has been introduced.

Of the latter it is hoped that the Lessons on Spiral and other Springs—which contain matter never, so far as I am aware, before published,—on Hydraulic and Electric problems connected with the Transmission of Power, and on Thermodynamics, will prove of special interest. The addition of fairly complete tables of "Moduli of Elasticity," "Ultimate Stresses," "Strength Moduli of Sections," &c., will supply the data necessary for working out any ordinary question connected with the subject.

In order to add to the completeness of each Lesson, and for convenience of reference, the answers to each set of Examples are placed at the end of the Lesson. The explanations and proofs have been given with as much clearness as possible, but in order to keep the book of a handy size the proofs are necessarily somewhat condensed. An attempt has been made to bring the book up to date, hence, whilst the ordinary British units are retained, considerable space has been devoted to an explanation of the Metric System, and examples have

been added for practice in converting from one system to the other.

Great care has been taken to approximate to accuracy in the answers, and I hereby tender my hearty thanks to those colleagues and students who have assisted me in this somewhat arduous part of the work.

The book embodies the results of many years' experience in teaching this subject, and I hope that in this newer and more complete form it may prove still more widely useful.

ROBERT GORDON BLAINE.

2, *Addison Road*, *Wanstead*, *Essex*,
August 5th, 1893.

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ELEMENTARY LESSONS,

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PRACTICAL MECHANICS AND MACHINE DESIGN.

LESSON I.

SUMMATION OF VECTOR QUANTITIES.

ANY quantity which is completely specified when its magnitude, direction, and sense or sign are given, is called a *vector* quantity. Such a quantity can be represented completely by a straight line, the *length* of the line representing the *magnitude* of the quantity, the *direction* of the line its *parallelism*, and an arrow-head on the line, or the sequence of two letters placed one at each end of the line, the third constituent to which the name *sense* has been given.

Quantities which require to be specified only in so far as regards their magnitudes can be represented by mere numbers, each with proper sign, and are called *scalar* quantities. Forces, velocities, accelerations, inductions, etc., are examples of vector quantities, whilst masses, volumes, capacities, etc., belong to the latter class.

The sum, or resultant, of any number of forces, or other vector quantities, acting at a point and in one plane, may be found *graphically* by drawing a polygon, the sides of which are straight lines parallel and proportional respectively to the forces, each side having an

arrow-head to denote the *sense* of the force which it represents, and these arrow-heads pointing concurrently round the figure. If the figure is closed, the forces are in equilibrium; if not, the closing side represents the sum or resultant required. If its arrow-head is concurrent with the others, it represents the *equilibrant*, and when reversed it represents the *resultant* of the other vectors. This is the easiest and best practical method of summing vectors. If the drawing is carefully done to a large scale, the result will probably be as accurate as the data from which, in practice, that result is obtained.

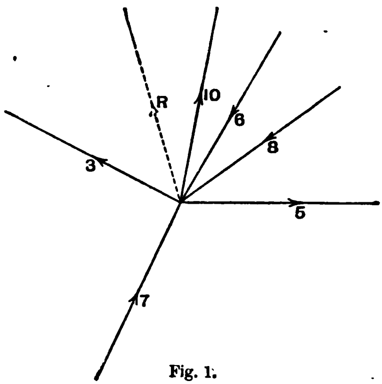


Fig. 1.

If there are only *two* forces, they may be represented by the adjacent sides of a parallelogram, the diagonal of which will represent the resultant. The forces must be arranged to act both *away from* or both *towards* the point before the parallelogram is constructed. If P and Q are two forces arranged in this way, θ the angle between them, and R their resultant, it is easy to show that $R^2 = P^2 + Q^2 + 2PQ \cos \theta$.

Except it is desirable to represent the resultant directly *in position*, a triangle is preferable to a parallelogram, the conditions being then the same as for a polygon.

It should be observed that in drawing the polygon the forces may be taken in *any* order, so long as each side of the polygon represents a particular force, and the arrow-heads point concurrently round the sides of the figure. Thus, in finding the resultant of the forces shown

in Fig. 1, they may be selected in such order as to give a polygon of the usual regular shape, or the polygon may be drawn as shown in Fig. 2.

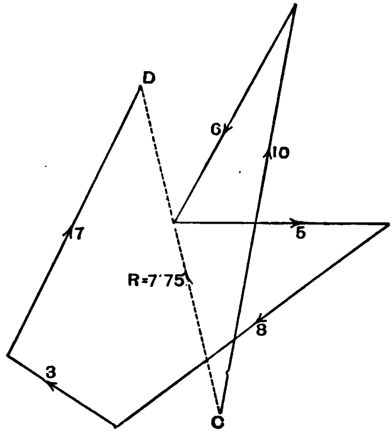


Fig. 2

There is another method of solving such questions, generally called the *analytical* method, which can, perhaps, be best illustrated to the beginner in the following way:—Let P T, P S, P R, P Q (Fig. 3), be drawn to represent to scale forces of 3, 6, 7, and 8 units acting at the point P. If these lines are

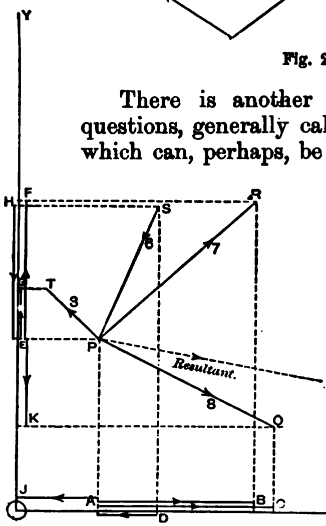


Fig. 3.

projected on two rectangular axes, O X and O Y, and if the algebraic sum of the horizontal projections be taken to form one side of a rectangle of which an adjacent side is the algebraic sum of the vertical projections of the same lines, then the diagonal of this rectangle represents the resultant sought. This is shown in Fig. 4, where N M is equal to A B + A C – A D – A J (Fig. 3), and N L is equal to E H + E K – E F – E G.

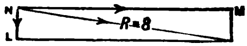


Fig. 4.

The horizontal and vertical projections referred to represent, respectively, the horizontal and vertical *components* of the given forces, and these components are usually found, not as here indicated, by actually drawing the projections, but by calculating them. It is probably well known to the student that each horizontal projection is equal to the length projected multiplied by the cosine of the angle that length or line makes with the horizontal axis, the vertical projection being found by multiplying by the *sine* of the same angle. The following question, fully worked out, will illustrate the method:—

Example.—Find the equilibrant of the forces represented in Fig. 5.

Let any force be denoted by F, and the *acute* angle

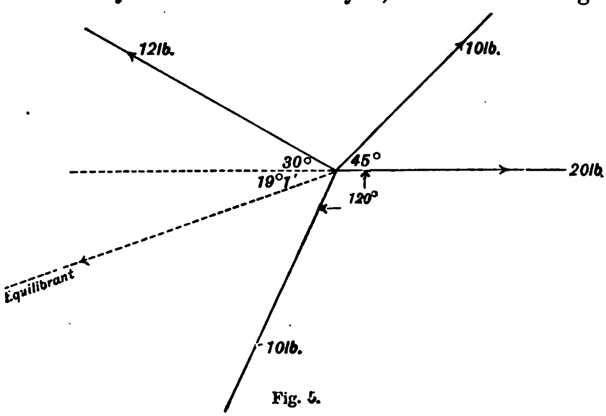


Fig. 5.

it makes with the horizontal by a. Let any horizontal component acting from left to right be considered positive, and any vertical component acting upwards also positive, those acting in the opposite directions being negative.

The following are the horizontal and vertical components :—

$F \cos a$.	$F \sin a$.
20	0
$10 \times \frac{1}{\sqrt{2}}$	$10 \times \frac{1}{\sqrt{2}}$
$-12 \times \frac{\sqrt{3}}{2}$	$12 \times \frac{1}{2}$
$-10 \times \frac{1}{2}$	$-10 \times \frac{\sqrt{3}}{2}$
$\Sigma F \cos a = 11{\cdot}68$.	$\Sigma F \sin a = 4{\cdot}41$.

By Euclid I. 47 the equilibrant or resultant

$$= \sqrt{(\Sigma F \cos a)^2 + (\Sigma F \sin a)^2}$$
$$= \sqrt{(11{\cdot}68)^2 + (4{\cdot}41)^2}$$
$$= \sqrt{136{\cdot}42 + 19{\cdot}45}$$
$$= 12{\cdot}80 \text{ lb.}$$

The angle this equilibrant makes with the horizontal is the angle whose tangent is (Fig. 6)

$$\frac{\Sigma F \sin a}{\Sigma F \cos a} = \theta \text{ say.}$$

$$\therefore \theta = 19^\circ\ 1' \text{ nearly.}$$

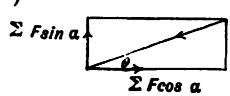


Fig. 6.

It should be noted that the symbol Σ means "the sum of all such terms as."

The equilibrant is represented by the diagonal of the rectangle in Fig. 6.

From the preceding it will readily be seen that the

condition of equilibrium of any number of forces or similar vector quantities acting at a point is (*graphically*) that the force polygon shall be closed, or (*analytically*) that the sum of the horizontal components shall be zero, and also that the sum of the vertical components shall be zero. If the forces do *not* act in one plane, the "graphic" condition still holds, but the sides of the polygon do not lie in one plane, and it is then called a *gauche* polygon.

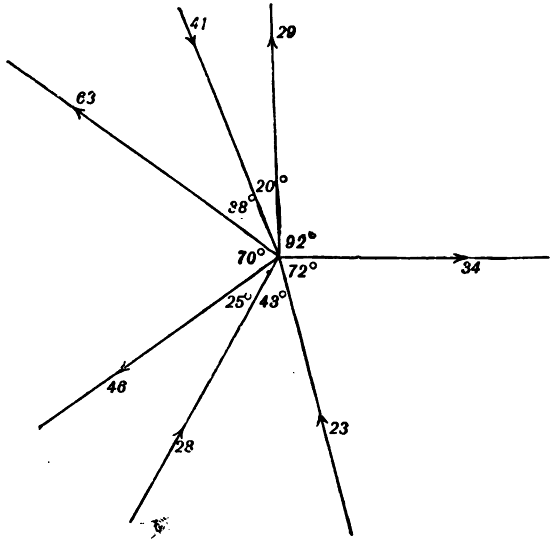


Fig. 7.

The forces projected on two planes at right angles to each other give a similar condition to that already stated, referred to those planes.

Forces acting in one plane, but not at one point, require the further "graphic" condition that the *link polygon* shall also be closed, or the further "analytical" condition that the sum of the moments of the forces about any assigned point shall be zero. These terms will be explained in a later lesson.

Numerical Examples.

1. A ship sails through the water at a uniform rate of 20 miles an hour, and a ball rolls across the deck at the same rate in a direction at right angles to the ship's course. Find the velocity of the ball relative to the water.
2. The wind blows from the north-east with a velocity of 20 miles an hour. Find the northerly and easterly components of its velocity.
3. A force of 100 lb.* acts at an angle of 60° to the horizontal. Resolve it into a vertical and a horizontal component.
4. Two equal pulls, each of 8 lb., act at a point. Find the angle between them if their resultant is 12 lb.
5. Two forces act at a point and away from it; they are in the ratio of 3 : 4, and their resultant is a mean proportional between them. Find the angle between their directions.
6. Forces of 2, 3, and $\sqrt{6}$ units are in equilibrium at a point. Find the angle between the directions of 2 and 3, the forces all acting towards the point.

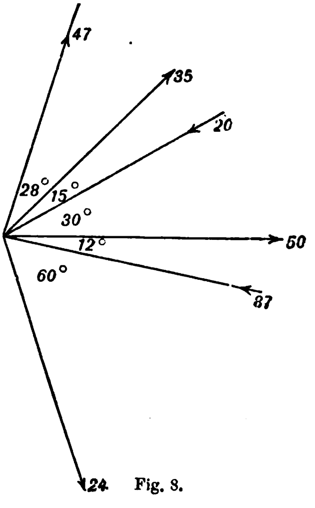


Fig. 8.

7. Find the resultant of velocities of 4, 5, and 6 feet per second acting at a point and towards it, the angles between 4 and 5 and between 5 and 6 being each 60°.

* For convenience the expression here and elsewhere employed is used instead of the more exact formula, "A force equal to the weight of 100 pounds," etc.

8. Find completely the resultant of the 7 forces represented in Fig. 7.
9. Find completely the equilibrant of the 6 forces represented in Fig. 8.

ANSWERS.

1. 28·28 miles an hour. 2. 14·14 miles an hour each.
3. 86·6 and 50 lb. respectively. 4. 82° 49′.
5. The angle whose cosine is $-\frac{13}{24}$.
6. The angle whose cosine is $-\frac{7}{12}$.
7. Resultant $= 10{\cdot}15$, and acts at an angle of $-9° \, 50'$ with the direction of 5.
8. The resultant is shown in Fig. 9.
9. The equilibrant is shown in Fig. 10.

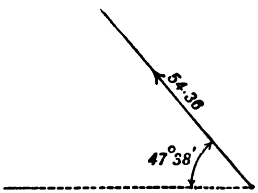


Fig. 9.

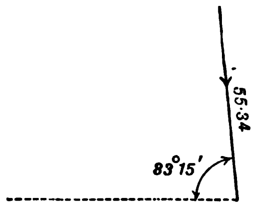


Fig. 10.

LESSON II.

THE USE OF SQUARED PAPER.

WHEN an experimenter has obtained numerical values of two quantities which are connected in such a way that any change in one is accompanied by a corresponding change of some kind in the other, his next care is generally to find out what the law of dependence of the one quantity on the other is, and to express that law in some way. Nowadays the connection or law is usually shown *graphically* by a curve drawn on squared paper. This has all the advantages of exhibiting to the eye a *map* or

picture of the law of variation ; the risk of serious error in getting out the law is reduced, errors of observation are to a great extent eliminated, and last, but not least —as a recent writer * puts it—" the student in drawing the curve is constantly on a voyage of discovery, and has

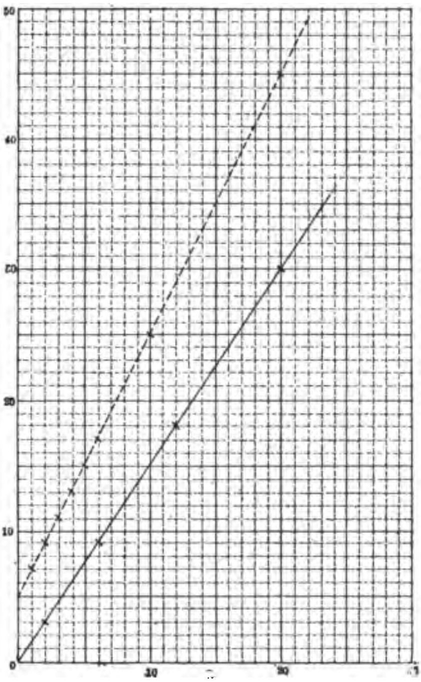


Fig. 11.

all the stimulus and pleasure of an original investigation."

A sheet of squared paper may be bought for a halfpenny or less ; it is crossed by a large number of equi-

* Dr. Wormell on "Plotting."

distant lines at right angles to each other, thus covering the paper with little squares. Usually every tenth line, both vertically and horizontally, is of a different colour from the rest, merely for the purpose of preventing confusion, and enabling the number of lines or squares to be readily reckoned. The lowest horizontal line on the paper is usually taken as one *axis* or line of reference, and the vertical line nearest the left-hand side of the paper as the other axis, except in the case of complicated curves, when the axes should be in the middle of the sheet. Distances along the horizontal axis represent values of one quantity, whilst distances along the vertical axis represent values of the other quantity, or *variable* plotted; these distances are measured by the number of little squares from the *origin* or point where the two axes intersect. Each little square may represent one or more units of the quantity, according to the scale to which the diagram is to be drawn. The distances, or quantities represented, go in pairs, each pair intersecting in a point of which those distances are called the co-ordinates, and for distinction the vertical one is usually called the *ordinate* of the point—denoted by the letter y—the horizontal distance, its abscissa, denoted by the letter x. When a number of pairs of values are thus plotted, a number of *points* are obtained, which when joined by a curve give the picture or trace of the law sought. Quite different scales may be adopted for the horizontal and vertical measurements, *i.e.*, the horizontal scale may be different from the vertical one.

Of course in plotting two definite quantities like the friction and load of a machine, the co-ordinates have definite meanings, and the symbols x and y are not required. The student will best grasp the meaning of these statements by going carefully through a few examples. Thus, plot the points whose abscissæ and ordinates are respectively (2, 3), (6, 9), (12, 18), (20, 30).

The points are shown plotted and joined by a straight line—the lower line in Fig. 11. It is evident that the law is in this case a very simple one, being $y = \frac{3}{2}x$.

This is an example of the simplest of all curves—a curve of the first degree—and passing through the origin.

The student should now plot the curves representing the following laws :—

(1) $y = 10$. (3) $y = x$.
(2) $x = 25$. (4) $y = 2x + 5$.
(5) $x = 16y^2$.

The best method to adopt is to tabulate in two columns corresponding values of x and y, found by giving to one of the variables—say y—any value, and finding the corresponding value of the other variable from the equation. The pairs of values thus obtained can then be plotted. The curve agreeing with (4) is shown by the dotted line in Fig. 11.

To find the *law* of such a line which has been obtained, say, by plotting experimental results. Take the upper curve in Fig. 11 as an example. The general law of such a curve is $y = a\,x + b$; a and b being numbers or constants, the first depending on the *slope* of the line, the last on the intercept it makes on the vertical axis.

To find a and b in this case. At one point on the line $x = 3$ and $y = 11$, at another point $x = 20$, $y = 45$.

Putting these values into the general law we get

$$\begin{array}{l} 11 = a \times \ \ 3 + b \\ 45 = a \times 20 + b \\ \hline 34 = a \times 17 \end{array} \quad \begin{array}{l} \\ \\ \text{subtracting, we get} \\ \text{or } a = 2. \end{array}$$

The value of a makes the first equation read

$$11 = 6 + b \quad \text{or} \quad b = 5;$$

hence the law is

$$y = 2x + 5.$$

If y represented friction and x represented load, the law would be

$$\text{Friction} = 2 \times \text{load} + 5.$$

Exercises.—Plot the curves whose laws are given below :—

(1) $x^2 + y^2 = 16$. (3) $y = 3x^3$.
(2) $x^2 - y^2 = \pm 100$. (4) $y = x^n$.

Plot the curves showing the relation of the following numbers :—

5. THE COMPOUND INTEREST LAW.—(*Interest added every instant.*)

TABLE SHOWING THE AMOUNT OF £100 INVESTED FOR *n* YEARS AT 5 °/₀ COMPOUND INTEREST.

No. of years = *n*.	Amount, A, = principal + interest	No. of years = *n*.	Amount, A, = principal + interest
1	105·2	8	149·1
2	110·5	9	156·8
3	116·1	10	164·8
4	122·1	11	173·3
5	128·4	12	182·3
6	135·0	13	191·5
7	141·9	14	201·4

6. THE CHARACTERISTIC CURVE OF A DYNAMO MACHINE.—The following results were obtained by experimenting with a small dynamo, having a gramme armature and being "series" wound. They show the way in which the *total* E.M.F. of the machine varies with the current that it gives out. Mean speed = 1843 revolutions per minute.

Total E.M.F.	Current.	*Total* E.M.F.	Current.
0 volts.	0 amperes.	8·44 volts.	11·25 amperes.
1·05 ,,	·75 ,,	8·907 ,,	12·00 ,,
2·614 ,,	2·25 ,,	9·366 ,,	13·5 ,,
4·196 ,,	3·75 ,,	10·12 ,,	15·75 ,,
5·376 ,,	5·25 ,,	10·754 ,,	18·00 ,,
6·35 ,,	6·75 ,,	11·21 ,,	20·25 ,,
7·14 ,,	8·25 ,,	11·382 ,,	22·5 ,,
7·812 ,,	9·75 ,,		

The above when plotted will give the characteristic curve of the machine.

LESSON III.

WORK, ENERGY, ETC.

THE British Engineer's unit of energy is the foot-pound—*i.e.*, the work necessary to raise one pound one foot high in London. The measure of a force in pounds, multiplied by the distance in feet through which it acts, gives the work done by the force in foot-pounds. For other units of force, etc., *see* page 41.

The "law of work" is simply this—that if we give 20 foot-pounds of energy to a machine, and there is no waste and no storage of energy, 20 foot-pounds, and *neither more nor less*, will be obtained from the machine. This is another way of putting the law known as the "conservation of energy," which states that "if a foot-pound of energy makes its appearance in any shape, one foot-pound at least in the same or some other form must have disappeared." This law directly affirms the impossibility of a "perpetual motion." The following example will explain the method of calculating amounts of work done, or of energy stored.

Example.—Find the work done in one minute by the steam on the piston of a steam-engine, the steam-pressure being supposed constant in the cylinder, and the area of the piston exposed to the steam-pressure the same on both sides.

The diameter of the piston is 12 inches, length of crank 12 inches, steam-pressure 40 lb. per square inch, speed 96 revolutions per minute.

Total pressure on piston $= \cdot 7854 \times 12^2 \times 40$ lb.*

Distance in feet travelled by the piston in one minute $= 4 \times 96$ feet.

Work done per minute $=$ force $\times$ distance $= \cdot 7854 \times 12^2 \times 40 \times 4 \times 96 = 1{,}738{,}000$ ft.-lb.

Numerical Examples.

1. How many units of work are expended in raising 1 cwt. from a depth of 60 fathoms? (1 fathom $=$ 6 ft.)

* The area of a circle is ·7854 times the square of its diameter, or 3·1416 times the square of its radius.

2. How many units of work must be expended in raising the materials for building a solid column of brickwork 100 feet high and 14 feet square? One cubic foot of brickwork weighs 112 lb.
3. How many units of work are spent in filling a tank with water; the tank measures 12 feet × 6 feet × $2\frac{1}{2}$ feet, the water has to be lifted an average distance of 20 feet, and a cubic foot of water weighs about 62·4 lb.?
4. Given that a man walking, and pushing or pulling, can do 3,130 foot-pounds of work in one minute, how many men would be required to raise—by means of a capstan—an anchor weighing 2 tons from a depth of 28 fathoms in 15 minutes, the friction of the machinery being neglected?
5. The mean section of a stream is 8 feet × 2 feet; its mean velocity 2 miles per hour; there is a fall of 12 feet on this stream. Find the number of foot-pounds of energy wasted per minute at this fall.

RATE OF FLOW.

IMPORTANT HINT.—Quantity = cross-sectional area × velocity. This is an *absolute* rule, and any units may be employed; but if square feet and feet per second are used in the right-hand side, the result will be in cubic feet per second. This is evident if we write the rule Q = A V. Let the units be as above, then

$$Q = A\ (\text{ft.}^2)\ V\left(\frac{\text{ft.}}{\text{sec.}}\right) = A\ V\ \frac{(\text{ft.})^3}{\text{sec.}}$$

a numerical quantity equal to the product of the numbers representing A and V respectively, the unit being one cubic foot per second $\left(\frac{\text{ft.}^3}{\text{sec.}}\right)$.

The student should make himself familiar with this rational method of treating formulæ, for the introduction of which we owe so much to Professors Oliver and A. Lodge. The method is, of course, applicable to any absolute formula given in this book, the units chosen being simply those most convenient for practical work.

See appendix to "Syllabus of Elementary Dynamics," by the Association for the Improvement of Geometrical Teaching.

ANSWERS.

1. 40,320.	2. 109,760,000.	3. 224,640.
4. 16 men.	5. 2,108,621 ft.-lb.	

LESSON IV.

VELOCITY RATIO AND MECHANICAL ADVANTAGE.

IF, by means of a machine, a force P be able to raise steadily a load W, the ratio of W to P is called the "mechanical advantage" or force ratio of the machine. It is not constant, and can only be determined accurately by experiment. If, however, we imagine the machine neither to waste nor store energy, then it is evident that if the ratio of the velocities of P and W is r to 1, $P \times r$ must be equal to $W \times 1$; the force ratio of the machine

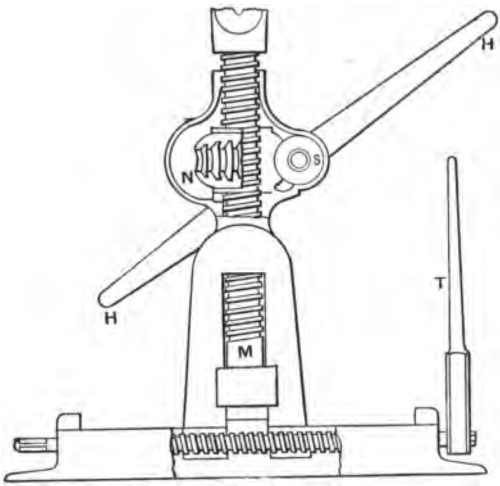


Fig. 12.

is therefore r to 1, or W *is to* P *inversely as their velocities.* If the velocity ratio of a machine is the ratio of the velocity of P to that of W, the *hypothetical* mechanical advantage of a machine is equal to its velocity ratio. The study of this "hypothetical" advantage is not of much practical importance, though usually taken up at length in the text-books. A state-

ment of the results obtained for some simple machines will be sufficient.

There are, however, two cases which are not usually taken up, to which a reference may not be out of place here.

Fig. 13.

Fig. 12 shows a screw-jack in which the handle H is not applied directly to the screw M, as is usually the case, but to a screw or worm S which works into a worm-wheel formed on the outside of the nut N of the

jack. N is prevented from moving axially in the direction of the screw M, and hence when N is turned once round the screw M advances by a distance equal to its pitch.

In the ordinary screw-jack this happens when the handle is turned once round; but here the handle has to be turned T times if there are T teeth on the worm-wheel. The velocity ratio is thus T times that of the ordinary jack of the same pitch and radius of handle.

Fig. 13 shows a recent form of hydraulic jack made by Messrs. Tangye, of Birmingham. When the plunger P is raised by the crank K, attached to the handle axis H, water is admitted through the valve V_1 in the plunger to the space under the plunger, whence it is forced, on the down stroke, through the valve V_2 to the space M under the ram R. As more and more water is forced into M the ram R is raised. The velocity ratio is easily determined, for if the plunger is, say, 1 square inch in section, the ram being 10 square inches, then if 10 inches in length of the plunger enter the water, 10 cubic inches of water are displaced to M, and R is raised 1 inch if all the spaces were filled with water, and if water is incompressible—which is very nearly true. We see, therefore, that the ratio of the velocities of plunger and ram is the ratio of the *cross-section of the ram to that of the plunger.* Since the impressed force is applied at the end of the handle, this ratio will require to be multiplied by the mechanical advantage of the handle to give the ratio of W to P. The above is given as an example of the method of finding the velocity ratio in almost all hydraulic machines.

The following are the velocity ratios of some simple machines :—

Common screw jack or press	Circumference of circle described by P $\div$ pitch of screw.
Hydraulic press or jack	Area of cross-section of ram $\div$ area of pump-plunger $\times$ mechanical advantage of pump-handle.
Pulley-blocks	Twice number of movable pulleys.
Differential pulley-blocks	Diameter of larger groove of top sheave $\div \frac{1}{2}$ difference of diameters of larger and smaller grooves.

Wheel and axle	Diameter of wheel ÷ diameter of axle.
Chinese windlass	Diameter of circle described by handle ÷ $\frac{1}{2}$ difference of diameters of larger and smaller parts of axle.
Inclined plane (force acting parallel to slope of plane).	Length of plane ÷ height.
Inclined plane (force acting parallel to base of plane).	Base of plane ÷ height.
Lever acted on by two forces in the same plane	Distance of smaller force from fulcrum ÷ distance of larger.

REAL MECHANICAL ADVANTAGE.

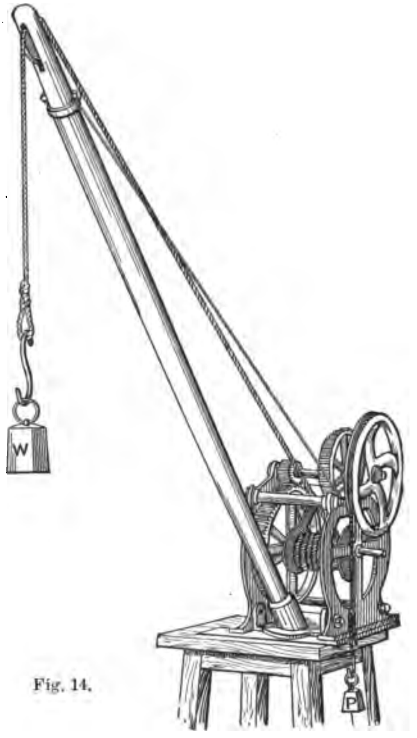


Fig. 14.

The actual mechanical advantage of a machine is not constant, and can be obtained, under given conditions, only by experiment. In closing the present lesson a brief explanation of such an experiment will now be given. The experiment in question is conducted on a small hand crane, such as is used for raising weights by manual labour. A view of the machine is shown in Fig. 14. For the purposes of the experiment the handle has been replaced by a pulley or wheel with a groove in the rim, round which a cord is wound, a weight, P, at the end of this cord raising a larger load, W, with a steady motion. The weight P represents the force which would have to be applied to the handle to raise the same load. Various loads are employed, and, by trial, the proper value of P—necessary to overcome friction and raise W steadily—is found. Some of the results obtained are shown in the following table:—

EFFICIENCY OF CRANE.

TABLE OF RESULTS (velocity-ratio $r = 18{\cdot}7$).

W = load raised in lb.	P = force applied to handle in lb.	Real Mechanical Advantage $= \frac{W}{P}$.	Efficiency $= \frac{W}{r \times P}$.
14	1·22	11·48	·61
28	2·125	13·18	·70
42	3·00	14·00	·75
56	3·91	14·32	·765
70	4·80	14·58	·77
84	5·71	14·71	·78
98	6·58	14·88	·79
112	7·52	14·90	·797
126	8·40	15·00	·80
140	9·30	15·05	·805
154	10·21	15·08	·807
168	11·08	15·16	·81

An illustration of the method of working out such examples as follow may be of service to the reader.

Example.—The diameter of the ram of an hydraulic jack is $1\frac{3}{4}$ inch, the diameter of the plunger $\frac{7}{8}$ inch, the mechanical advantage of the handle 10. Find the total mechanical advantage. If the experimental law of the

machine is $P = \cdot025\ W + 5\cdot3$ (P being the force, in pounds, necessary to raise steadily a load W), find its efficiency when lifting $\frac{1}{2}$ ton.

$$\text{Velocity ratio or mechanical advantage} = \frac{\text{area of ram}}{\text{area of plunger}} \times \text{mechanical advantage of handle.}$$

$$= \frac{\cdot7854 \times (1\cdot75)^2}{\cdot7854 \times (\cdot875)^2} \times \text{by } 10 = 40.$$

$$\text{Efficiency} = \frac{\text{work given out}}{\text{work put in}} = \frac{W}{P \times r},\ r \text{ being the velocity ratio.}$$

$$= \frac{1{,}120}{40\,(\cdot025 \times 1{,}120 + 5\cdot3)} = \frac{1{,}120}{1{,}332} = \cdot84, \text{ or } 84 \text{ per cent.}$$

NOTE.—It may not be out of place to explain that the term "counter-efficiency," mentioned in many books, means the reciprocal of the efficiency,

$$= \frac{\text{useful work} + \text{wasted work}}{\text{useful work}}$$

$$= 1 + \frac{\text{wasted work}}{\text{useful work}}$$

$$= 1 + e,$$

the letter e being used to denote the ratio $\frac{\text{wasted work}}{\text{useful work}}$.

Numerical Examples.

1. A plane rises 1 in 20. Find the pull, in the direction of the plane, necessary to move a load of 1 ton, friction being neglected.
2. The pitch of a screw-jack is $\frac{1}{2}$ inch, the distance from the axis of the screw to the end of the handle 26 inches. Find the velocity ratio. If the law (in pounds) is $P = \cdot03\ W + 9\cdot45$, find the load which will be lifted by a force of 56 lb. applied at the end of the handle. Find also the efficiency of the jack for this load.
3. The diameter of the larger sheave of a differential pulley-block is 9 inches, that of the smaller sheave 8·8 inches. Find the velocity ratio. If the law of

the machine is $P = \cdot012\ W + 6\cdot38$ (in lb.), what weight will be lifted by a force of 56 lb.? Find the efficiency.

4. The pitch of the propeller of a steamship is 20 feet, its velocity 80 revolutions per minute, and slip 10 per cent. Find the speed of vessel in knots (nautical miles per hour).
5. In an hydraulic press the mechanical advantage of the handle is 10, the diameter of the ram 11 inches, and the diameter of the pump-plunger $\frac{3}{4}$ inch. Find the total mechanical advantage.
6. In a screw-jack the pitch of the screw is $\frac{3}{8}''$; the radius of the circle described by the handle is $18''$. Find the hypothetical mechanical advantage.
7. Find the mechanical advantage, neglecting friction, of a Chinese windlass, the radius of the handle being 18 inches, and the diameters of the two parts of the axle 9 and $8\frac{1}{2}$ inches respectively.
8. In the last example what would have been the mechanical advantage if the axle had been uniform and of 8 inches diameter, there being no movable pulley.
9. Find the efficiency of a machine the law of which (in pounds) is $P = 4\cdot21 + \cdot0429\ W$, when lifting 1 ton, the velocity ratio being 40.
10. The law of a certain machine is $P = \cdot048\ W + 1\cdot6$, and its efficiency when lifting a load of 10 cwt. is 65 per cent. Find the velocity ratio. Same units as before.
11. Plot the curve and find the law connecting P and W in the results for the crane given on page 19. Plot columns 1 and 3, and also 1 and 4.

ANSWERS.

1. 112 lb.
2. 326·7 to 1. 1,551·66 lb. ·0848, or $8\frac{1}{2}$ per cent. nearly.
3. 90 to 1. 4,135 lb. 82 per cent.
4. 14·2.
5. 2,151·1.
6. 301·59.
7. 144 to 1.
8. $4\frac{1}{2}$ to 1.
9. 55·83 per cent.
10. Velocity ratio 31·1.
11. $P = \cdot064\ W + \cdot32$.

LESSON V.

MOMENTS OF FORCES.

THE law of moments is as follows:—*If a number of forces act on a body, tending to turn it about an axis, there will be equilibrium if the sum of the moments of the forces tending to turn it against the hands of a watch is equal to the sum of the moments of the forces tending to turn it with the hands of a watch;* or, in other words,

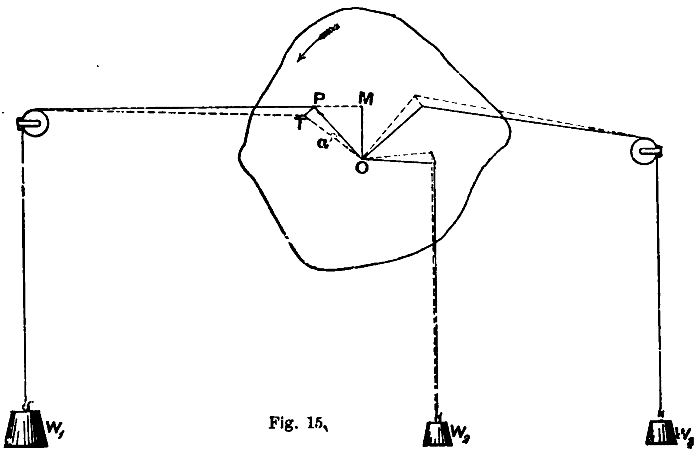


Fig. 15.

if the algebraic sum of the moments is zero. The "moment" of a force about a point is measured by the product of the force and its perpendicular distance from the point. A moment is a vector quantity, the axis of the moment being taken as the line of reference.

This law may be proved, for forces acting in one plane, by a simple application of the law of work.

In doing so we may adopt a method which is often used in cases where motion alters the configuration of forces and distances—viz., that of supposing a very small motion given to the body. Let the small angle turned

through by the body, which is acted on by forces all in one plane, and which is pivoted so that its weight may be neglected, be called a (Fig. 15); all lines on the body will turn through the same angle. Fig. 16 is an enlarged drawing of a part of Fig. 15, and it will be seen from it that the work done by W_1 is $W_1 \times PQ$. But since O M is perpendicular to P M, and O P perpendicular to P T, the two triangles O M P and T Q P have the angles at O and P equal, and the angles at Q and M also equal being right angles, hence the triangles are similar, therefore $\frac{QP}{PT} = \frac{OM}{OP}$.

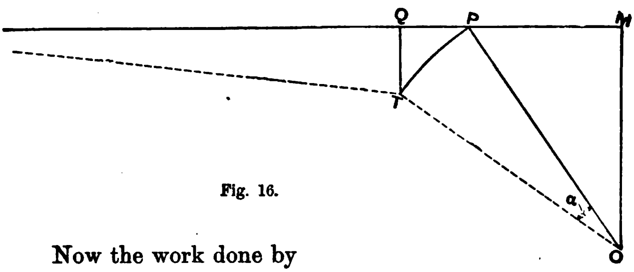


Fig. 16.

Now the work done by

$$W_1 = W_1 \times PQ = W_1 \times \frac{PQ}{PT} \times PT$$

$$= \frac{W_1 \times OM \times PT}{OP}.$$

But the angle $a = \frac{PT}{OP}$ (in radians).

$\therefore$ the work done by $W_1 = W_1 \times OM \times a$.

Similarly the work on W_2 can be shown to be equal to $W_2 \times$ its perpendicular $\times a$, and so on for W_3, etc.; whence, dividing across by a, we get from the law of work the rule,

$$W_1 \times \text{its perpendicular} = W_2 \times \text{its perpendicular} + W_3 \times \text{its perpendicular};$$

or the algebraic sum of the moments of all the forces must be zero.

This law may be employed in solving a great number of useful examples. It may be applied to the forces acting on a lever; to find the centre of gravity—or, more properly, *centre of mass*—of a body; to find the *centre of area* of any given plane area, and to numerous similar cases. As this last may present some difficulty to the beginner, the following example is given fully worked out.

Example.—A uniform circular disc 6 inches in diameter has a circular hole 2 inches in diameter punched out of it,

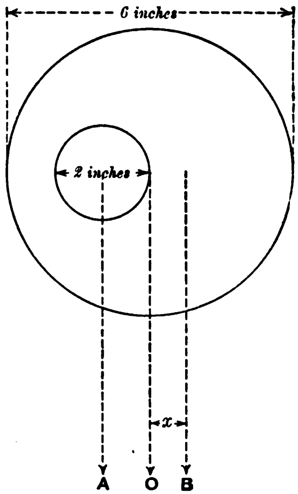


Fig. 17.

as shown in Fig. 17, the edge of the hole passing through the centre of the disc. Find the centre of area of the remainder of the disc.

Let x inches be the distance of the new centre of area from the centre of the plate, then if we imagine the hole to be again filled by the piece cut out of it, there will be equilibrium about the centre of the completed disc. Taking moments about that centre, and remembering that the weight of each part is simply proportional

to its area, and for our purpose may be taken *as* that area, we have—

$$\cdot 7854 \times 2^2 \times 1 = \cdot 7854\ (6^2 - 2^2) \times x,$$
$$\text{or } 2^2 = (6^2 - 2^2)\ x,$$
$$4 = (36 - 4)\ x;$$
$$\therefore x = \frac{4}{32} = \tfrac{1}{8}.$$

i.e., the centre of area has moved over $\frac{1}{8}$ of an inch.

Another very useful class of question is that in which the *two* equilibrants of a number of given forces are required. The following exercise will make this plain:—

Example.—A bar, whose weight is neglected, is

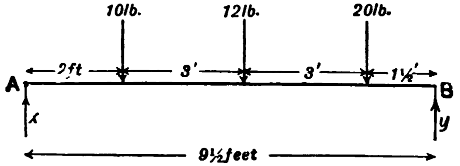


Fig. 18.

loaded as shown in Fig. 18. Find the two supporting forces x and y.

Taking moments round an axis at A,

$$10 \times 2 + 12 \times 5 + 20 \times 8 - y \times 9{\cdot}5 = 0.$$

Whence $y = 25{\cdot}26$ lb. But $x + y = 42$, $\therefore x = 16{\cdot}74$.

The question can also be readily solved graphically by the method referred to at page 133.

Numerical Examples.

1. A uniform rod 8 feet long is supported at a point 3 feet from one end, and loaded at this end with a weight of 10 lb. Find the weight at the other end which will produce balance, the weight of the rod being neglected. If the weight of the rod is 3 lb., what is the correct answer?
2. A uniform lever, 12 feet long, balances about a point 2 feet from one end when loaded at that end with a weight of 30 lb. Find the weight of the lever.
3. A rigid bar, whose weight may be neglected, is supported at points 12 feet apart and on the same

level, and is loaded by the following vertical loads —viz., 5 lb., 8 lb., 6 lb., 11 lb., and 6 lb., at points 1 foot, 3 feet, 5 feet, 8 feet, and 10·5 feet respectively from one of the supports. Find the supporting forces. If it were necessary to balance this bar on a single support, find the position of that support.

4. A bar similar to the last, supported at points 18 feet apart, is acted on by vertical loads of 16, 20, 18, and 12 lb., at points 2, 5, 9, and 13 feet respectively from one of the supports. Find the supporting

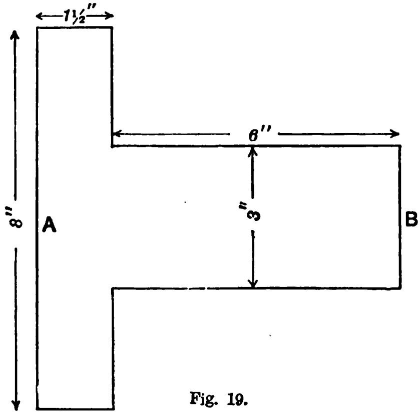


Fig. 19.

forces, and find also the position of a *single* equilibrant of the given forces.

5. Find the centre of area of the figure shown in Fig. 19.

6. A circular hole 1 inch in diameter is punched out of a uniform circular disc 6 inches in diameter, the edge of the hole being 1 inch from the circumference of the disc. Find the centre of area of the remainder.

7. A uniform plate is of the shape of an isosceles triangle, the base being 6 inches, and each of the equal sides 8 inches long. If a square be cut out of this triangle, one side of the square space coinciding with

the base of the triangle, and two of its corners on the two equal sides, find the length of a side of the square, and the centre of area of the portion of the triangle that remains.

8. A rigid bar, whose weight may be neglected, is acted

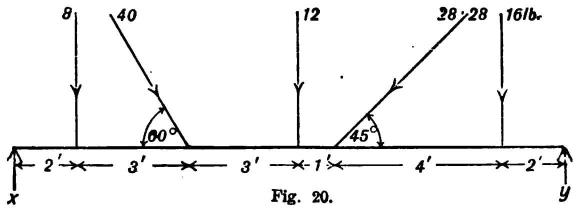


Fig. 20.

on by the forces shown in Fig. 20. Find the magnitude and direction of the supporting forces at x and y.

9. A lever safety-valve has the following dimensions:—mean diameter of valve 3″, weight of valve 8 lb., distance of centre of valve from fulcrum $2\frac{1}{2}$″, weight of lever 16 lb., distance of its centre of gravity from fulcrum 13″. Find where a load of 35 lb. must be hung from this lever so that the steam may blow off at 85 lb. pressure per square inch.

10. In the sketch of a weigh-bridge (Fig. 21), find the

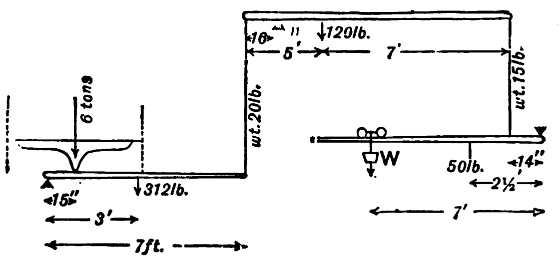


Fig. 21

magnitude of the weight W, so that there may be equilibrium.

NOTE.—The pull in each vertical rod at the top end will be equal to that at the bottom end *plus the weight of the rod.*

11. A wall of brickwork, 2 feet thick and 20 feet high, sustains on the inner edge of its summit a certain pressure on every foot of its length, the direction of the pressure being inclined at 60° to the horizon. Find its amount when it is on the point of overturning the wall. A cubic foot of brickwork weighs 112 lb.
12. Weights respectively of 33, 41, and 56 lb. are placed at the angles A, B, C of a right-angled isosceles triangle. Prove that the distance of the centre of gravity of the weights from the right angle B is equal to half of one of the equal sides.

ANSWERS.

1. 6 lb. 5·4 lb. 2. 15 lb.
3. 18·5 and 17·5 lb. 5·83 feet from the 18·5 lb. force.
4. 41 and 26 lb. 6·8 feet from the 41 lb. force.
5. $4\frac{1}{3}$ inches from B.
6. ·043 inch from the centre of the disc, and on the side farthest from the hole.
7. Side of square, 3·316 inches. Centre of area, 3·001 inches from the base of the triangle.
8. $x = 45·76$ lb., $y = 44·88$ lb., both vertical.
9. 36·4 inches. 10. W = 25·97 lb.
11. 541·86 lb. per foot length of wall.

LESSON VI.

POWER, EFFICIENCY, ETC.

POWER is the rate of doing work. 33,000 foot-pounds of work done in one minute is called one horse-power (which we will denote by the symbol "HP"). This is the mechanical unit of power. The electrical unit of power is one *watt,* being the power developed in a circuit in

which a current of one *ampere* flows with a pressure of one *volt.* There are 746 watts in one HP. Watts = amperes × volts. To find the HP of any machine is simply to find the work done by it per minute, and divide by 33,000; or to find the work done by it per second, and divide by 550.

There are many useful examples of finding the power developed by various agents and machines which might be referred to, but few of them present much difficulty. There is one class of question which is very important, and which may require explanation—viz., the estimation of the power that may be obtained from a stream or river with a given fall. The first thing to be done is to obtain the quantity, or weight, of water flowing over the fall per unit of time. There are many methods of doing this, such as by calculation of the mean area of the stream and the mean velocity of the water, which may be obtained approximately by means of floats or current-meters. The mean area multiplied by the mean velocity gives the quantity passing per unit of time. These methods are only roughly approximate. The most accurate method, for small or moderately-sized streams, is that of the weir-gauge or gauge notch, Professor James Thomson's being the best. In this case the water is allowed to flow through a V-shaped notch cut in a board, the notch being of the shape of a right-angled isosceles triangle with sharp edges. The only measurement required is the height h of still-water level above the lowest point or angle of the notch (Fig. 22).

If this height is measured in feet, then the quantity of water, in cubic feet per second, passing over the notch, is obtained by *raising the number expressing this height to the fifth power, extracting the square root and multiplying the result by* 2·645. If the notch is rectangular and of length L feet, with sharp edges, as before, the quantity Q is obtained from the rule

$$Q = 3{\cdot}33\ (L - {\cdot}2\ h)\ h^{\frac{3}{2}}.$$

This is known as the Lowell formula, deduced by Mr. Francis from experiments carried out at Lowell, in

Massachusetts, U.S.A. The following exercises in calculating the power of machines, etc., will be readily understood, if the first example be carefully followed.

Example.—Find the HP involved in raising 20 tons through a height of 80 feet in one hour.

$$\text{Work done per minute} = \frac{20 \times 2{,}240 \times 80}{60} \text{ ft.-lb.}$$

$$\therefore \text{ HP} = \frac{8 \times 20 \times 2{,}240}{6 \times 33{,}000} = 1{\cdot}81.$$

NOTE.—To measure HP electrically, multiply the current in amperes by the E.M.F. in volts, and divide by 746.

The efficiency of a machine is the ratio of the power

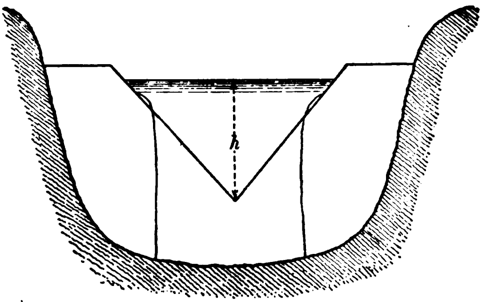


Fig. 22.

it gives out to the power given to it. Hence, if the useful HP of a machine is 20, and its efficiency is 60 per cent., there must be 33·33 HP given to it in order to produce this useful effect.

1. What is the HP wasted in example 5 of Lesson III.?
2. What HP is equivalent to the lifting of 50,000 cubic feet of water 100 feet high in 24 hours?
3. 300 tons of coal are to be lifted from a depth of 200 fathoms by an engine and machinery, the combined efficiency of which is 57 per cent. What will be the indicated HP of the engine if it do this work in 10 hours? (The term "indicated HP" is explained in the next lesson.)

4. An engine of 20 indicated HP is employed to pump water from a marsh; the area of the marsh is half a square mile, and the water has to be lifted 10 feet. If the combined efficiency of the engine and pump is $40\frac{1}{2}$ per cent., how long will it take the engine to lower the surface of the water six inches? (It is supposed that there is no loss of water by evaporation, etc., and that the marsh does not receive water during this time.)
5. The efficiency of a turbine water-wheel is 70 per cent., and 300 cubic feet of water pass through it per minute. What fall is necessary in order that we may obtain 10 HP from the turbine?
6. A turbine giving out 20 HP is employed to drive the dynamo which generates the current for an electric tramway. If the combined efficiency of the dynamo, leads, motors, etc., is 30·4 per cent., and the weight of a train is 5 tons (resistances to motion being 20 lb. per ton), at what rate will the train move?
7. If a gas-engine of 16 indicated HP were employed to drive the dynamo in the last example, and if the locomotive car, weighing 3 tons, runs alone at 8 miles an hour, what is the efficiency of the whole arrangement, including engine mechanism, dynamo, leads, motors, etc.—resistances being 25 lb. per ton?
8. Professor James Thomson's **V**-shaped weir-gauge is employed to measure the flow of water in a certain stream; the vertical distance from still-water level to the lowest point of the notch is observed to be 1·3 feet. The water passing through this gauge, with a fall of 20 feet, is employed to drive a water-wheel of 60 per cent. efficiency. What HP may be obtained from the wheel?
9. A weir-gauge, with a rectangular notch, is employed to measure the flow of water in a certain stream, the length of the notch being 3 feet, and the vertical distance from still-water level to the sill of the notch being 1·28 feet. Suppose this water, with a fall of 15 feet, is employed to drive an overshot

water-wheel of 55 per cent. efficiency, find the useful HP of the wheel.

10. The current for the Portrush Electric Tramway is generated by a dynamo driven by two turbines, the fall of water available being 26 feet. If the efficiency of each turbine is 70 per cent., and that of the dynamo and connecting machinery 80 per cent., and if the dynamo gives out a current of 100 amperes with an E.M.F. of 250 volts, find the quantity of water passing through the turbines per minute.

11. A waterfall is to be utilised for electric lighting. The engineer sent to inspect the place finds out the following data:—The water at one place flows in a straight rectangular channel 4 feet wide, the water being 2 feet deep and its average velocity 2 feet per second. The available fall is 20 feet, and the water-wheels to be used have an efficiency of 60 per cent., the dynamo efficiency being 80 per cent. Neglecting all other losses of energy, find approximately how many 60-watt incandescent lamps may be supplied.

12. Suppose the stream were much smaller than in the last example, and the flow measured by Prof. James Thomson's V-shaped weir-gauge, h being 1·5 feet, how many lamps could be supplied under similar conditions to those given in the last question?

ANSWERS.

1. 63·89 HP. 2. 6·55 HP. 3. 71·45 HP.
4. 11·29 days of 24 hours. 5. 25·22 feet.
6. 22·8 miles an hour. 7. 10 per cent.
8. 6·94 HP. 9. 12·37 HP.
10. 1,217·2 cubic feet per minute.
11. About 216 lamps. 12. 100 lamps.

* (Page 33.) This sum can best be obtained by marking off the lengths consecutively on a strip of paper and measuring the whole length thus marked.

LESSON VII.

INDICATED HORSE-POWER, ETC.

WORK, being measured by the product of force and distance, can be represented by an area. This is readily seen if the force is constant, as the area is then a rectangle; but it is not difficult to prove that, even if the force is variable, the area included by a curve, whose ordinate, at any point, represents the value of the force when at a distance represented by the abscissa of the point, is proportional to the work done by the variable force.

An indicator diagram, such as that shown in Fig. 23,

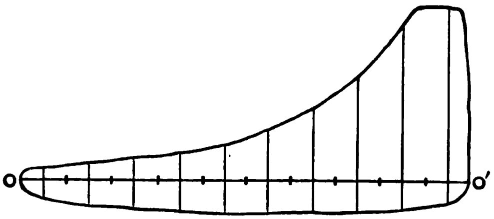


Fig. 23.

taken from a steam-engine, for instance, is a figure of the kind referred to, its area representing the work done by the steam on the piston of the engine whilst it was being traced out. It is usual in calculating the indicated HP of an engine from the indicator diagram to find the *mean effective pressure of the steam* by finding the mean height of the diagram—either by dividing the area of the diagram, as found by a planimeter, by the length O O′; or by dividing the sum* of the ten ordinates shown in the figure by 10—and multiplying the number representing this height by the number of pounds' pressure per square inch required to move the pencil of the indicator vertically on the diagram one inch. This mean pressure *multiplied by the area of the piston in square inches, by the mean velocity of the piston in feet per minute,*

* See footnote on previous page.

and divided by 33,000, gives the indicated HP, if the engine is of the usual double-acting kind. If it is single-acting, the power is half that obtained by the above rule; or if the mean piston velocity be taken as the *average distance the piston moves per minute under the action of pressure from the steam, burning gas, or other working fluid*, then the rule is applicable to all engines. This should be carefully noted.

Example.—The mean height of an indicator diagram —taken from an engine with 10-inch piston, 12-inch crank, speed 150 revolutions per minute—is 0·9 inch, the scale of pressure being 32 lb. per square inch to one inch. Find the indicated HP of the engine.

Mean pressure of steam $= 32 \times \cdot 9 = 28{\cdot}8$ lb. per square inch.
Dictance travelled by piston per minute under this pressure $= 4 \times 150$ feet.

$$\therefore \text{ Indicated HP} = \frac{\cdot 7854 \times 10^2 \times 28{\cdot}8 \times 4 \times 150}{33{,}000} = 41{\cdot}127.$$

Numerical Examples.

1. The mean pressure of the steam in the cylinder of a certain steam-engine is 26 lb. per square inch, the diameter of the cylinder of the engine is 12 inches, length of crank 12 inches, and the speed 96 revolutions per minute. Find the indicated HP of the engine.
2. The mean height of an indicator diagram is $1\frac{1}{2}$ inch, the scale of the diagram being such that an ordinate one inch long represents 30 lb. pressure per square inch; if the diagram is taken from the same engine as that mentioned in the last example, going at the same speed, find the indicated HP.
3. An indicator diagram has an area of 4 square inches, the vertical scale of the diagram being the same as in the last example, and the horizontal scale being such that one inch on the diagram represents one foot travel of the piston of the engine. If the piston is 9 inches in diameter, and the speed 96 revolutions per minute, find the indicated HP of the engine.
4. A very good pumping-engine consumes 2 lb. of coal

per hour per indicated HP; an engine of this class has a piston of $3\frac{1}{2}$ feet diameter, 6 feet stroke, mean pressure of steam in cylinder 40 lb. per square inch, and the engine makes 48 working strokes per minute: find the indicated HP and the weight of coal required for one day of 10 hours.

5. A certain pumping engine, whose mechanical efficiency is 85 per cent., can, with pumping machinery of 45 per cent. efficiency, lift 4,000 cubic feet of water per hour from a depth of 150 fathoms. If the consumption of coal per hour per HP be the same as in the last example, find the weight of coal required for a working day of 10 hours.

6. If 1 lb. of coal gives out in burning, in the most perfect manner, 7,500 Centigrade heat-units, find the efficiency, including boiler, furnace, etc., of the engines in the last two examples.

   One Centigrade unit of heat is, in round numbers, equivalent to 1,400 ft.-lb., and one Fahrenheit unit of heat to 778* ft.-lb., of mechanical energy.

7. In an indicator diagram, such as that shown in Fig. 23, the sum of the ten ordinates, drawn as in the figure, was $9\frac{7}{16}$ inches, the scale of the diagram being 32 lb. per square inch to one inch. If the crank of the engine is 14 inches long, the piston 13 inches in diameter, and its rod $1\frac{7}{8}$ inch in diameter, the speed of the engine being 100 revolutions per minute, find the indicated HP.

ANSWERS.

1. 34·217 HP. 2. 59·22 HP.

3. 44·42 HP. Let x = length of diagram in inches.
y = mean height of diagram in inches.
Then $x\,y = 4$, as given.
$30\,y$ = mean pressure in lb. per sq. inch.
x = travel of piston in feet (for one stroke).

* Recent determinations by Messrs. Griffiths and Clark, of Cambridge, give the numbers 1402 and 778·99 respectively. Professor Rowland and others give nearly the same values.

$\therefore$ $30\,x\,y$ = mean pressure $\times$ travel of piston.

$\cdot 7854 \times 81 \times 30\,x\,y$ = mean total pressure $\times$ travel of piston for one stroke.

$\cdot 7854 \times 81 \times 30 \times 4 \times 2$ = work done per revolution.

$\therefore$ $\cdot 7854 \times 81 \times 30 \times 4 \times 96 \times 2$ = work done in 96 revolutions = 33,000 HP.

$$\frac{\cdot 7854 \times 81 \times 30 \times 4 \times 96 \times 2}{33{,}000} = \text{HP}.$$

$\therefore$ HP = 44·42.—*Ans.*

4. 483·6 HP. 4·3 tons. 5. 2·64 tons.
6. ·094 or 9·4 per cent. 7. 55·5 HP.

LESSON VIII.

POWER FOR TRACTIVE PURPOSES.

THE work required to be expended in drawing a carriage up an incline A C (Fig. 24) consists of two parts: that necessary to overcome the ordinary tractional resistances, as on the level, through a distance A C, and that required to lift the weight through a vertical height B C. The term "gradient" usually means the ratio of B C to A C, B C being taken as unity. Thus a gradient of 1 in 200 means a vertical rise of 1 foot to every 200 feet measured along the slope. In calculations relating to the *power* necessary to draw a vehicle of given weight up a given slope, it is only necessary to find x, the distance the vehicle will go in one minute, then add to the work done in moving over a distance x on a level road the work done in lifting it through the vertical height B C.

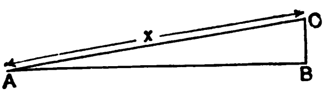


Fig. 24.

Example.—Find the power necessary to draw a train weighing 100 tons up an incline of 1 in 200 at the steady rate of 30 miles an hour, the resistance to motion on a level road at the same rate being 12 lb. per ton.

Here $B\,C = \dfrac{1 \times x}{200}$.

x = distance train goes in 1 minute $= \dfrac{30 \times 5{,}280}{60} = 2{,}640$ feet.

Hence $B\,C = 13{\cdot}2$ feet.

Resistance to motion on level $= 12 \times 100$ lb.

$\therefore$ work done per minute against ordinary frictional resistances $= 1{,}200 \times 2{,}640$ ft.-lb. approximately.

Also work done in lifting the train through the height B C $= 100 \times 2{,}240 \times 13{\cdot}2$ ft.-lb.

Hence the total work done per minute
$= 1{,}200 \times 2{,}640 + 100 \times 2{,}240 \times 13{\cdot}2$ ft.-lb. $= 6{,}124{,}800$ ft.-lb.

But work done per minute $\div$ 33,000 = HP.

$$\therefore \text{HP expended} = \frac{6{,}124{,}800}{33{,}000} = 185{\cdot}6.$$

Numerical Examples.

1. Find the useful HP of an engine which draws a train weighing 120 tons, on the level, at 40 miles an hour. (Resistances, when not given, to be found from the formula—resistance in lb. per ton $= \dfrac{(\text{speed in miles per hour})^2}{171} + 8$).

2. What is the HP necessary to draw a train, weighing 120 tons, up an incline of 1 in 130, at the rate of 35 miles an hour?

3. If a locomotive move a train, weighing 100 tons, at a steady speed of 40 miles an hour on the level, at what rate would it draw the same train up an incline of 1 in 120 whilst giving out the same useful HP, the resistance due to friction being assumed equal in the two cases?

4. Find the indicated HP of a locomotive which draws a train weighing 150 tons up an incline of 1 in 200 at 33 miles per hour, the efficiency of the mechanism of the engine being taken as 80 per cent.

5. A locomotive engine has two cylinders, each 12 inches in diameter, the cranks being 12 inches in length, and making 115 revolutions per minute. If the mean pressure of the steam in each cylinder is 90 lb. per square inch, at what rate will this engine draw

a train, weighing 200 tons, up an incline of 1 in 90; its useful HP being 75 per cent. of its indicated HP, and tractive resistances at the same speed on the level being taken at 12 lb. per ton.

6. The indicated HP of a locomotive is 400, and the efficiency of the mechanism of the engine 75 per cent. If this engine draw a train weighing 200 tons up an incline of 1 in 300 at a steady speed, find that speed, the tractive resistances averaging 10 lb. per ton.

ANSWERS.

1. 222·08 HP. 2. 362·76 HP.
3. 19·29 miles an hour. 4. 421·9 HP.
5. Useful HP 212·8. Speed 10·78 miles an hour.
6. 32·3 miles an hour.

LESSON IX.

FORCE, MASS, AND VELOCITY. UNIFORM ACCELERATION. MOMENTUM AND KINETIC ENERGY.

Velocity is rate of motion, and if uniform—in which case equal distances are passed over in equal intervals of time—it is measured by the distance passed over in one second.

Acceleration is rate of change of velocity, and if uniform—in which case there is an equal gain or loss of velocity in equal times—it is measured by the velocity gained or lost per second. When a body moves with a continually increasing speed, its acceleration is said to be *positive*; if, on the other hand, the velocity diminishes, the acceleration is *negative*. We shall here deal only with *uniform* velocity and acceleration. From the statement made above it will readily be seen that if a body move with a uniform velocity v over a distance s in time t, v must be equal to $\frac{s}{t}$, or $s = v\,t$.

The laws of uniform acceleration can be readily

illustrated graphically. Thus in Fig. 25 the ordinate A E represents the body's velocity at the beginning, and B D represents its velocity at the end of the interval of time, t, considered.

Since acceleration, a, is the *velocity gained* in unit time, evidently that gained in time, t, is t times as much, or $a\,t$; this is represented by C D. The distance passed over by the body in the time considered is $\frac{1}{2}(u + v)t$,

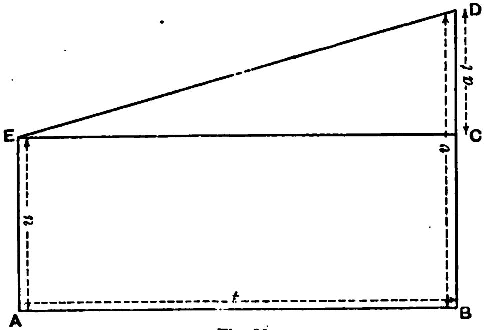


Fig. 25.

or is the *average velocity* multiplied by the time *; this distance, s, being represented by the area of the figure A B D E, which is equal to the area of the rectangle A B C E + the area of the triangle E C D.† Putting these last two statements into algebraic form we have

$$(1)\ v = u + at$$
$$(2)\ s = ut + \tfrac{1}{2}at^2.$$

Eliminating the term t from these two equations we get a third

$$(3)\ v^2 = u^2 + 2as.$$

It must be understood that in these rules the + sign is replaced by − if the acceleration is *negative*. If, in a particular case, the initial velocity is zero, that is to say, if

* This is not so self-evident as it at first sight appears to be. It is, however, usually assumed to be true.

† It is only the scalar magnitude of the distance that is here represented.

the body *starts* at the beginning of the interval of time considered, then the rules become

$$(1)\ v = at$$
$$(2)\ s = \tfrac{1}{2}at^2$$
$$(3)\ v^2 = 2as.$$

A body falling freely under the action of gravity has a uniformly accelerated motion, if we neglect atmospheric friction, or suppose it to produce a constant effect. The body in this case *gains* a velocity of about 32 feet per second *every second* of its motion, or, in dynamical language, the acceleration is 32 feet per second per second ($32\ \frac{\text{ft.}}{\text{sec.}^2}$). This acceleration is usually denoted by the letter g; it varies slightly with the latitude of the place, and also with the height above sea-level. If, then, a body is allowed to fall freely from rest, through a height h, or for a time t, the rules become

$$(1)\ v = gt$$
$$(2)\ h = \tfrac{1}{2}gt^2$$
$$(3)\ v^2 = 2gh.$$

Force, Mass, and Acceleration.

The connections between force, mass, and acceleration are illustrated by Atwood's machine, which consists of a pulley, mounted in as frictionless a manner as possible, and bearing a cord to the two ends of which two equal weights A and B are fastened. An additional small weight P, in the form of a long thin plate, is added to one of the equal weights, which, when let go, move with a uniformly accelerated motion, till at a certain point in its fall the little weight P is lifted off in passing through a ring; A and B now moving on with a uniform *velocity*. This arrangement enables the *acceleration* of a given mass—that of A + B + P—due to a given force, P, to be measured.

By varying, first, the force, and then the mass, it is found that (1) acceleration is proportional to force when mass is constant; (2) acceleration is inversely proportional to mass when the force is constant; and hence acceleration is proportional to $\frac{\text{force}}{\text{mass}}$.

We have now only to choose such units as shall make this proportion simple equality.

There are three systems of units, any of which will do this.

UNITS.

In the C.G.S. (centimetre, gramme, second) system the unit of length is one centimetre, which is $\frac{1}{100}$th part of one metre—a metre being about 39·37 inches—the unit of mass is one gramme, which represents the mass of one cubic centimetre of distilled water at the temperature of 4° C., and the unit of force is one *dyne*, or that force which *acting on one gramme for one second generates a velocity of one centimetre per second.*

This system is very scientifically arranged as regards the connection between its various units, and the decimal system being employed, such operations as multiplication or division are rendered extremely easy. It has the great practical disadvantage that its units of force and mass are inconveniently small, and hence ordinary forces with which practical men have to deal require to be expressed by very large numbers. Gravitation units are also employed in connection with this system, the weight of one gramme or one kilogramme (1,000 grammes) being used as a unit of force.

In the British Absolute System the unit of length is one foot, the unit of mass is one pound, and the unit of force is *one poundal*, or that force which acting on one pound for one second generates a velocity of one foot per second. In this system the unit of force is about $\frac{1}{32}$ of the force with which the earth attracts one pound-weight, and however excellent and scientific the system may be, practical men find the unit of force a little too small; however, it may in time be generally adopted.

In the *British Gravitation* or *Engineer's* system the *force due to the weight of one pound* is taken as the unit of force; but as it varies a little at different localities and levels, it is usual now in the best books to find this unit defined as the force with which the earth attracts a pound-weight *at the sea-level at Greenwich.* The unit of

mass will then be the mass of about 32·2 pounds; the unit of length being, as before, one foot or $\frac{1}{3}$ of the standard yard, which is the distance between the centres of two gold plugs in a platinum bar kept at the Standards Office of the Board of Trade at Westminster. In any of these systems the law, acceleration $= \frac{\text{force}}{\text{mass}}$, or force $=$ mass $\times$ acceleration, is true.

The last-mentioned system is that adopted by most British engineers, and is used throughout these lessons. For convenience we shall use such expressions as "a force of 10 pounds," meaning a force equal to the weight of 10 pounds—the symbol "lb." being employed simply to denote "pound" or "pounds," without reference to whether force or mass is intended. The mass of any body is, in this system, obtained by *dividing its weight in pounds by* 32·2.

The unit of work in each system is the work done by unit force acting through unit distance, and called the *erg*, the *foot-poundal*, and the *foot-pound* respectively.

Momentum.

We have seen that if a force act on a body free to move, the resulting motion will be governed by the law—

$$\text{force} = \text{mass} \times \text{acceleration},$$

or, in algebraic form, $F = m \times a$. Going back to the beginning of the present lesson, we had the elementary law $v = u + a\,t$, the $+$ sign being replaced by a $-$ if the velocity *decreases*.

Taking these two rules together, we have

$$F = a \times m,$$
$$\text{and } v = u + at.$$

From the last

$$a = \frac{v - u}{t};$$

hence the first becomes

$$F = \left(\frac{v - u}{t}\right) m \quad \text{or } Ft = vm - um,$$

the same double meaning being attached to the sign on the right-hand side as before. The product of the mass and velocity of a body is called its *momentum;* it was formerly called "quantity of motion," and hence the law just obtained agrees with Newton's second law of motion, "change of motion is proportional to the impressed force, and takes place in the direction in which that force acts." Force is therefore *momentum per second.* If the time t is too short to be measured, the force's "impulse" is measured by the momentum produced or destroyed.

Kinetic Energy.

The kinetic energy of a moving body expressed in foot-pounds is equal to the *product of half its momentum and its velocity,* the velocity being measured in feet per second. Suppose a body of mass m to move a distance of s feet under the action of, and in the same direction as, a constant force F, then the work done on the body —moving with a uniformly accelerated motion—is $F \times s$. Let v_1 be its velocity at the beginning, and v_2 its velocity at the end of the interval of time considered. Then, since distance = average velocity × time, $s = \frac{(v_2 + v_1)}{2} . t$, and from what has been already stated in regard to the value of a force in connection with change of momentum, we have

$$F = m \frac{(v_2 - v_1)}{t},$$

$$\text{hence } F \times s = m \frac{(v_2 - v_1)}{t} \times \frac{(v_2 + v_1)}{2} . t$$

$$= \frac{m}{2}(v_2^2 - v_1^2) = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2,$$

or the *work done is equal to the change of kinetic energy.* If the force is of the nature of a resistance and *opposes* the body's motion, the rule is still true, only v_1 will then be greater than v_2, *negative* work being done on the body by the force F.

Numerical Examples.

1. A boy drops a stone down the shaft of a mine and

finds that it takes 3·5 seconds to reach the bottom: what is the depth of the shaft?

Here the rule is

$$h = \tfrac{1}{2}gt^2,$$

in this case

$$\begin{aligned} h &= \tfrac{1}{2} \times 32{\cdot}2 \times 3{\cdot}5^2 \\ &= 16{\cdot}1 \times 12{\cdot}25 \\ &= 197{\cdot}2 \text{ feet.} \end{aligned}$$

(2. How long will a mass weighing 150 lb., and moving with a velocity of 20 feet per second, move against an opposing force of 10 lb.?

Let t be the required time,

$$\text{then } Ft = mv,$$

$$10 \times t = \frac{150}{32{\cdot}2} \times 20,$$

$$\text{or } t = 9{\cdot}3 \text{ seconds.}$$

3. A shot weighing 12 lb. leaves the mouth of a gun with a velocity of 1000 feet per second: find its kinetic energy, and the mean resistance offered by an obstacle into which it penetrates a distance of 2 feet.

$$\begin{aligned} \text{Kinetic energy} &= \tfrac{1}{2} \text{ mass} \times (\text{velocity})^2 \\ &= \frac{1}{2} \times \frac{12}{32{\cdot}2} \times 1000^2 = 186{,}335 \text{ ft.-lb.} \end{aligned}$$

Let F be the average resisting force; then—

$$186{,}335 = F \times 2, \text{ or } F = 93{,}167\tfrac{1}{2} \text{ lb.}$$

4. Find the velocity of the stone in Example 1 just before it touched the bottom of the shaft, and also its velocity when it had fallen 100 feet.

5. A stone is projected vertically upwards with a velocity of 50 feet per second: how high will it rise?

6. A man descends a mine 4000 feet deep with a uniform velocity. Having descended for 4 minutes, he drops a stone, which reaches the bottom in 10 seconds. Find the velocity with which the man descends.

× 7. What is the acceleration produced by a force of 10 lb. acting on a mass which weighs 100 lb.?

8. If a locomotive move a train which, including engine,

weighs 100 tons, giving to it a uniformly accelerated motion such that in 10 minutes after starting it has a velocity of 20 miles an hour, find the weight of a second train which the same engine draws, and which gets up a speed of 7 miles an hour in 5 minutes; the total resistance and the pull of the locomotive being taken as the same in both cases.

9. In Atwood's machine equal weights of $\frac{2}{3}$ lb. are suspended by the string passing over the pulley, and a bar weighing $\frac{1}{16}$ lb. is laid across one of them. This bar, after falling 2 feet, is lifted off. How far will the remaining masses move in the next 10 seconds?

10. A man weighing 160 lb. descends the shaft of a mine with an acceleration of 2 feet per second per second: find the pressure he exerts on the floor of the cage.

✓11. A ball is laid on a *smooth* inclined plane, the inclination of which is 1 in 20: find the acceleration of the ball, and the distance it will go in 3 seconds.

NOTE.—The acceleration in such a case is $g \times \sin a$, where a is the angle the plane makes with the horizontal.

12. A train weighing 100 tons, moving at the rate of 30 miles an hour, is brought to rest in $2\frac{1}{4}$ minutes: find the average resultant force acting against the train's motion.

13. A body weighing 20 lb. falls freely through a height of 200 feet: find the average force required to stop it in the next 5 seconds.

14. A pile-driver weighing 300 lb. falls through a height of 20 feet, and is stopped in $\frac{1}{5}$th of a second: find the average force it exerts on the pile.

15. A ball weighing 8 lb., and moving with a velocity of 50 feet per second, receives a blow in a direction at right angles to its line of motion, and it then proceeds in a direction making an angle of 45° with the direction of its former path: find the "impulse" imparted by the blow.

16. A railway waggon weighing 10 tons is drawn from

rest by a horse; after going 300 feet it is moving at the rate of 5 feet per second, if the tractive resistances amount to 8 lb. per ton: find the amount of work done by the horse.

17. A projectile leaves the mouth of a gun with a velocity of 1000 feet per second: find its velocity when it is at a height of 100 feet above the level of the gun, neglecting atmospheric friction.

    NOTE.—This question can be solved from the fact that the total store of energy the projectile possesses is supposed to remain constant. Just as it leaves the mouth of the gun its energy is all kinetic and known in amount; at the higher point it is partly kinetic and partly potential, their sum being the same total amount as before. The part consisting of potential energy is known from the height, hence the kinetic energy, and therefore the velocity, of the projectile is easily found. The weight of the projectile is not required.

18. If the angle of projection in the last example were 30° with the horizontal, find the greatest height to which the projectile would rise, and its velocity at its highest point.

19. What will be the range or distance the projectile will go, on the horizontal plane, through the mouth of the gun?

    NOTE.—The range is found by multiplying the horizontal component of the initial velocity by the time of flight. The time of flight is found by considering that the time taken by the projectile to go to its highest point (*i.e.*, half the time required) is the same as the time a body must fall in order to acquire a velocity equal to the vertical component of the velocity of projection.

20. Find the work done by a winding-engine which raises a cage weighing 6 tons from a pit 400 yards deep, and gives to the cage a velocity of 24 feet per second at the top of the shaft. If this work is done in 6 minutes, find the HP actually applied.

21. If the weights in Atwood's machine are 3 and 2 lb., find the work done by gravity whilst the heavier falls 10 feet.

22. A train moving at 15 miles an hour comes to the foot of an incline of 1 in 300. If the frictional

resistances amount to 8 lb. per ton, how far will the train go if the steam is shut off?

23. A railway carriage weighing 5 tons is started down an incline of 1 in 100. What will be its velocity after it has gone 200 yards? Resistances as in the last example. Also how far must it run before it acquires a velocity of 60 miles an hour?

24. A pendulum-bob weighing 20 lb. is suspended by a wire, the length from the point of suspension to the centre of the bob being 16 feet. The pendulum swings 30° from the vertical on each side. Find its potential energy in its highest position, and also its velocity when passing its lowest position.

ANSWERS.

4. 112·7 feet per second. 80·25 feet per second.
5. 38·8 feet. 6. 9·56 feet per second is the man's velocity.
7. 3·22 feet per second per second. 8. 143 tons nearly.
9. 24 feet nearly. 10. 150·1 lb.
11. 1·61 feet per second per second. 7·245 feet.
12. 2267·3 lb.
13. 34·09 lb. (20 lb. of this is required to balance the body's weight).
14. 1667·2 lb. 15. $12\frac{1}{2}$. 16. 32,695·6 ft.-lb.
17. 996·77 feet per second. 18. 3882·1 feet. 866 feet per second.
19. 26,896 feet (a little over 5 miles).
20. 16,248,208 ft.-lb. 82·06 HP. 21. 10 ft.-lb.
22. 1,088·5 feet. 23. 10·7 miles an hour. 18,705 feet.
24. 42·88 ft.-lb. 11·75 feet per second.

LESSON X.

COMPARISON OF C.G.S. AND OTHER UNITS.

FORCE.

1 lb. (*gravitation*) $= 4{\cdot}45 \times 10^5$ dynes (*C.G.S. absolute*)
1 gram $= 981$ dynes.
1 poundal $= 13825$ dynes.
1 poundal $= \frac{1}{32{\cdot}2}$ lb. in Britain.

Length.

1 foot	= 30·4797 centimetres.
1 metre	= 39·37 inches.
1 inch	= 2·54 cm.
1 mile	= 160,933 cm.
1 nautical mile	= 185,230 cm.

Area.

1 square inch = 6·45 square cm.
1 square foot = 929·01 square cm.

Mass.

1 lb. = 453·59 grams.
1 kilogramme = 2·2 lb.

Stress (*Gravitation Measures*).

1 lb. per sq. foot = ·48826 gm. per sq. cm.
1 lb. per sq. inch = 70·31 gm. per sq. cm.

Work (*Gravitation*).

1 foot-pound = 13,825 gm. cm.

Work (*Absolute*).

1 gm.-cm.	= 981 ergs.
1 ft.-lb.	= $1·356 \times 10^7$ ergs.
1 ft.-poundal	= 421390 ergs.

Power (*Gravitation*).

1 HP	= $7·604 \times 10^6$ gr.-cm. per second.
1 force-de-cheval or cheval-vapeur	= $7·5 \times 10^6$ gr.-cm. per second.

Power (*Absolute*).

1 HP = $7·46 \times 10^9$ ergs per sec.
1 watt = 10^7 ergs per sec.

Numerical Examples.

1. Express the velocity of 1 knot in cm. per second.
2. A piece of material will stand the stress of 10,400 lb. per square inch, find this in gms. per sq. cm.; also dynes per sq. cm.
3. The value of E for wrought-iron is 29×10^6 lb. per sq. inch, find this in dynes, and in gms. per sq. cm.
4. An agent can do 3,000 ft.-lb. of work in 1 minute, find the rate of working in cm. dynes (ergs) per second.

5. An engine gives out 12 HP, find this in watts, chevaux-vapeur, kilogrammetres per second, and ergs per second.

1 kilogrammetre	$= 9{\cdot}81 \times 10^7$ ergs.
1 HP	$=$ 550 ft.-lb. per sec.
	$=$ 746 watts.
1 cheval-vapeur	$=$ 542·48 ft.-lb. per sec.
	$=$ 736 watts.
	$=$ 75 kilogrammetres per sec.

6. A train moves at 30 miles per hour, find its velocity in feet per second, in kilometres per second, and in cm. per second.

At a bend in a river the water moves at 5 miles an hour, the radius of the lines of flow being 9,100 cm. Find the slope of the surface of the water, taking $g = 981$ cm. per sec. per sec.

The C F (*see* page 187) of a mass m grams, rotating at velocity of v cm. per second, in a circle of r cm. radius, is $= \dfrac{m\,v^2}{r}$ dynes.

8. Young's modulus for brass is (Everett) $9{\cdot}86 \times 10^{11}$ dynes per square cm., find it in lb. per square inch.
9. Taking as unit of heat the heat required to raise one cubic centimetre of water 1° C., find the value, *in ergs*, of Joule's mechanical equivalent of heat, being given that the heat required to raise one lb. of water 1° F. is equivalent to 779 ft.-lb.

ANSWERS.

1. 51·45 cm. per sec.
2. 731,224 gm. per sq. cm. 717,519,347 dynes per sq. cm.
3. $20{\cdot}007 \times 10^{11}$ dynes per sq. cm. $203{\cdot}9 \times 10^7$ gm. per sq. cm.
4. 678×10^6 cm.-dynes (ergs) per sec.
5. 8,952 watts. 12·163 chevaux-vapeur. 912·48 kilogrammetres per sec. $89{\cdot}51 \times 10^9$ ergs per sec.
6. ·0134 kilometres per sec. 1341 cm. per sec.
7. Slope, 1 in 178. 8 $14{\cdot}29 \times 10^6$. 9. $42{\cdot}9 \times 10^6$.

LESSON XI.

KINETIC ENERGY OF ROTATING BODIES.

THE expression for the kinetic energy of a rotating body is obtained by taking the sum of the kinetic energies of all the little masses into which we can conceive the body as being divided. All little masses, at different distances from the axis of rotation, have different linear velocities, but all have the same *angular* velocity. If this angular velocity is A radians per second, the kinetic energy of each little portion, whose mass is m, and whose distance from the axis is r, is $\frac{1}{2} m \mathrm{A}^2 r^2$ (v being $= \mathrm{A}\, r$). The sum of all such expressions gives the kinetic energy of the whole body, or

$$\Sigma \tfrac{1}{2} m \mathrm{A}^2 r^2 = \text{the kinetic energy} = \tfrac{1}{2} \mathrm{A}^2 \Sigma m r^2 = \tfrac{1}{2} \mathrm{A}^2 \mathrm{I},$$

where I is the moment of inertia of the body about the axis of rotation, *being the sum of all such products as each little mass and the square of its distance from the axis.* Engineers prefer to express the angular velocity of a rotating body in *number of revolutions per minute*, and the expression for this is obtained as follows:—

$$\text{Kinetic energy} = \tfrac{1}{2} \mathrm{I}\, \mathrm{A}^2, \text{ but } \mathrm{A} = \frac{2 \pi n}{60},$$

where n is the number of revolutions per minute, hence

$$\tfrac{1}{2} \mathrm{I}\, \mathrm{A}^2 = \tfrac{1}{2} \mathrm{I} \frac{4 \pi^2 n^2}{3{,}600} = \frac{\pi^2 \mathrm{I}}{1{,}800} \times n^2.$$

The moment of inertia of the body about the same axis is constant, hence if M * is a constant $= \frac{\pi^2 \mathrm{I}}{1{,}800}$, the kinetic energy of the rotating body can be expressed in the convenient form $\mathrm{M}n^2$.

The constant M may be obtained by experiment in cases where the calculation of the moment of inertia would be difficult and tedious.

It is not necessary even to experiment with the same body as that whose store of kinetic energy, at a given

* This method of putting the rule is due to Professor Perry.

speed, we wish to know. The experiment may be performed with an exactly similar, but much smaller or larger body; and the rule, *the M's*—or moments of inertia—*of two similar bodies rotating about similarly placed axes are as the fifth powers of their like linear dimensions*, may be employed to find the M of the other.

This will readily be seen from Fig. 26, where two bodies are shown, one s times the other in all corresponding linear dimensions. A little mass m in the smaller

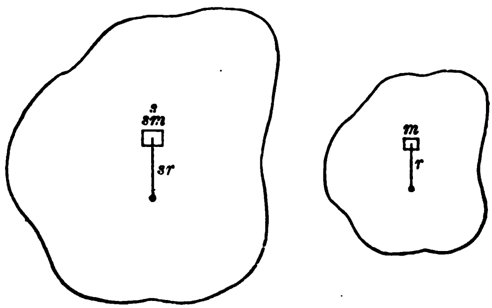


Fig. 26.

corresponds to a mass s^3m in the larger, and the moment of inertia of the smaller body is $\Sigma\, m\, r^2$, whilst that of the larger is $\Sigma s^3 m\ (sr)^2 = s^5\ \Sigma\ m\ r^2$. Hence the moments of inertia (or M's) are as the fifth powers of their like linear dimensions.

The rules given in this lesson are especially useful in designing fly-wheels for machines. The moments of inertia of some regular solids are given in the Appendix, page 198.

Example.—A fly-wheel is required to give out 12,000 ft.-lb. of energy whilst its speed decreases from 151 to 149 revolutions per minute. Find its M and also its diameter if similar to one the M of which is ·02, and diameter 2 feet.

It is evident that the energy given out is the energy at the higher speed *minus* the energy at the lower speed.

$$\therefore\ 12{,}000 = M\,(151^2 - 149^2) = M \times 300 \times 2$$

$$\therefore\ M = \frac{12{,}000}{2 \times 300} = 20.$$

$$\text{and } \frac{M}{\cdot 02} = \frac{D^5}{2^5}\text{, whence } D = \sqrt[5]{\left(\frac{20}{\cdot 02}\right)} \times 2.$$

$\therefore$ D = 7·96 feet, the diameter of the wheel required.

Numerical Examples.

1. There is a fly-wheel the weight of whose rim is 10 tons, and its mean radius 5 feet. Calculate its M approximately. The moment of inertia may be taken as approximately equal to the mass of the rim × the square of its mean radius in feet.
2. If the energy taken from a fly-wheel, whilst its speed lowers from 150 to 140 revolutions per minute, is 15,000 ft.-lb.; find the M of the fly-wheel. Find also its diameter if it is similar to another wheel the M of which is ·0024, and diameter 1·25 feet.
3. The M of a fly-wheel is 2·456; the mean radius of its rim is 2·6 feet. Calculate the weight of the rim approximately.
4. Find the diameter of a fly-wheel which is to be similar to an existing wheel, the M of which is ·0024, and diameter 2 feet; the new wheel never to run at more than 96 or less than 90 revolutions per minute, and to give out during a certain interval 20,000 ft.-lb. more than it receives.
5. We want a fly-wheel to contain a store of 1,000 ft.-lb. of energy when rotating at 20 revolutions per minute, and to be similar to an existing wheel which is 4 feet in diameter, and contains a store cf 1,350 ft.-lb. when rotating 30 times per minute. Find the M and diameter of the new wheel.
6. A grindstone 6 feet in diameter and 9 inches broad rotates 75 times per minute. Find its M, and also the kinetic energy it possesses at this speed, its specific gravity being 2·14. If the axle of the stone is 2 inches in diameter, how long will friction take to stop it, the coefficient of friction being supposed

constant for all speeds and equal to ·089? The M of a cylinder whose length and diameter are l and d feet respectively is $w\ l\ d^4 \div 59{,}800$, where w is the weight in lb. of one cubic foot of the material.

7. In non-condensing steam-engines, the steam being cut off at from $\frac{1}{2}$ to $\frac{1}{5}$ of the stroke, it is found that the fluctuation of energy is from $\frac{1}{6}$ to $\frac{1}{4}$ of the work of one revolution. In an engine of this class, of 25 IHP, the cut-off being $\frac{1}{5}$ of the stroke, and the limiting speeds 129 and 131 revolutions per minute, find the M of its fly-wheel. Find also the diameter of its fly-wheel if similar to that whose dimensions are given in Example 2.
8. For condensing engines, cut off from $\frac{1}{3}$ to $\frac{1}{8}$ of the stroke, the fluctuation of energy is from $\frac{1}{6}$ to $\frac{1}{5}$ of the work of one revolution. A steam-engine of this class of 50 IHP, cut off at $\frac{1}{8}$; limits of speed 70·4 and 69·6 revolutions per minute. Find the M and diameter of its fly-wheel if similar to that in Example 2.
9. In an Otto gas-engine the fly-wheel receives energy only during $\frac{1}{4}$ of one complete cycle (2 revolutions). Hence the fly-wheel must store, approximately, the energy required from the engine during the remaining $\frac{3}{4}$ of the cycle. An engine of this class, of 16 indicated HP, drives a dynamo machine, the limiting speeds being 149·5 and 150·5 revolutions per minute. Find the M of its fly-wheel, and also its diameter if similar to that mentioned in Example 2.
10. The fly-wheel of a certain traction engine is 5 feet in diameter, and is, let us suppose, similar to that given in Example 2. Suppose the steam shut off when the fly-wheel is rotating at 150 revolutions per minute, how far will the engine move on the level before coming to rest; the engine and truck weighing 30 tons, resistances to motion averaging 20 lb. per ton, and the efficiency of the engine mechanism being 65 per cent.?

11. Suppose the earth to be a sphere of 4,000 miles radius, mean specific gravity 5·5, find approximately its kinetic energy due to its rotation on its axis. The M of a sphere is $w\ d^5 \div 112{,}126$.
12. Find the weight of fly-wheel rim per HP, which at a linear velocity of 70 feet per second will store the work of 5 revolutions of the engine, the mean speed being 100 revolutions per minute.

ANSWERS.

1. M = 95·36 approximately.
2. M = 5·17. Diameter, 5·8 feet.
3. Weight = 2135·1 lb. 4. Diameter, 11·9 feet.
5. M = 2·5. Diameter = 4·43 feet.
6. M = 2·167. Energy = 12,189·37 ft.-lb. Time, 2½ minutes.
7. M = 3·05. Diameter, 5·23 feet.
8. M = 42·09. Diameter, 8·825 feet.
9. M = 17·6. Diameter, 7·413 feet. 10. 59·9 feet.
11. $19{,}771 \times 10^{25}$ ft.-lb. approximately.
12. 21·45 lb. per HP.

LESSON XII.

FRICTION AND STRENGTH OF BELTS.

THE rules on which calculations as to the strength of belts are founded are (1) $\frac{(\mathrm{N}-\mathrm{M})v}{33{,}000} = \mathrm{HP}$ transmitted, and (2) $\frac{\mathrm{N}}{\mathrm{M}} = \varepsilon^{\mu\theta}$, where N and M are the pulls in the belt on the tight and slack sides respectively, μ is the coefficient of friction between belt and pulley, θ the angle embraced by the belt, expressed in *radians*, v the velocity of the belt in feet per minute, and ε the base of the Napierian system of logarithms = 2·71828 . . . . .

Rule (1) is easily proved by supposing a pulley turned by a belt from the ends of which two unequal weights, N and M, are suspended. The work given to the pulley in one minute is evidently the difference of

that done *by* the greater weight in falling and that done *on* the smaller weight in raising it, or, in other words, equal to $(N - M)\ v$, which, divided by 33,000, gives the HP.

A short investigation of rule (2) may be of interest to students. Let A B be a very small portion of the pulley touched by the belt P Q (Fig. 27); consider the

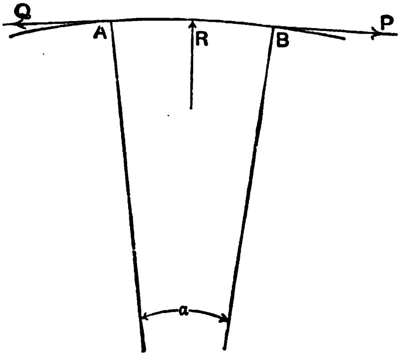


Fig. 27.

equilibrium of this small portion of the belt. We see that the forces acting upon it are the pulls P and Q and the normal reaction R. Constructing the triangle of forces (Fig. 28), we have—since a is a *very* small angle—

$$R = P\ a.$$

But friction $= \mu \times$ normal pressure.

$$\therefore \text{ friction} = \mu\ R = \mu\ P\ a,$$

$$\text{or } P - Q = \mu\ P\ a.$$

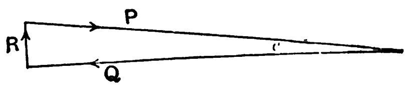


Fig. 28.

Now let $P - Q$ become indefinitely small; a becomes $d\ \theta$,

$$\text{and } d\ P = \mu\ P\ d\ \theta.$$

$$\text{or } \frac{d\ P}{P} = \mu\ d\ \theta.$$

Integrating between the limits M and N, and o and θ respectively, we get

$$\text{Log. N} - \text{log. M} = \mu\, \theta,$$

$$\text{(2) Hence } \frac{N}{M} = \epsilon^{\mu\theta}.$$

Combining the two formulæ (1) and (2), and eliminating M, we get

$$N = \frac{33{,}000 \text{ HP}}{v} \left(\frac{\epsilon^{\mu\theta}}{\epsilon^{\mu\theta} - 1} \right).$$

Let b be the breadth of the belt and t its thickness in inches, 330 lb. per square inch the safe stress for leather belting, then we get the rule from which calculations are made:—

$$b \times t = \frac{100 \text{ HP}}{v} \left(\frac{\epsilon^{\mu\theta}}{\epsilon^{\mu\theta} - 1} \right) \ldots . (3).$$

Assuming that the lapping on the smaller of the two pulleys is generally about $\frac{2}{5}$ of the circumference, and that $\mu = \cdot 3$, we get a very simple approximate rule, which may be used in many cases, viz. :—

$$b \times t = \frac{200 \text{ HP}}{v} \ldots . (4).$$

Rule for speeds : $D\,N = d\,n$ . . . . (5),

D being the diameter of the larger pulley.
N „ „ speed „ „ „
d „ „ diameter „ smaller „
n „ „ speed „ „ „

Numerical Examples.

1. Fig. 29 shows the arrangement of pulleys and belts used for driving a dynamo machine F, from the steam-engine A.

   If the speed of A is 96 revolutions per minute, find the speed of F, assuming that there is no slipping of the belts.

2. A machine is driven from a pulley 4 feet in diameter by means of a belt. If the difference of pull in the two sides of the belt be 20 lb., and the pulley makes 150 revolutions per minute, find the HP transmitted by the belt.

3. The fly-wheel of a steam-engine is 9 feet in diameter, and makes 96 revolutions per minute. What must be the difference of pull in the two sides of the belt if 26 HP be transmitted by it?
4. A dynamo machine receives 20 HP, its pulley is 10 inches in diameter, and revolves 800 times per minute. Find approximately the breadth of single belt $\frac{1}{4}$ inch thick which will drive it.
5. A belt transmits 8 HP, and travels at a velocity of

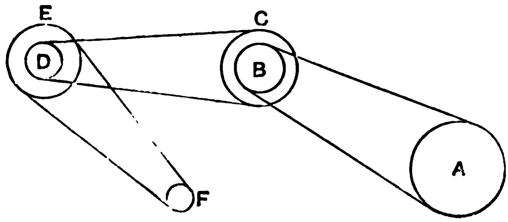


Fig. 29.

Diameter of A = 57″, B = 36″, C = 42″, D = 24″, E = 48″, F = 14″.

20 feet per second. Find approximately the breadth of single belt $\frac{7}{32}$ inch thick which is necessary.
6. A line of counter-shafting revolves 150 times per minute, and a pulley on it 2 feet in diameter gives 2 HP to the pulley of a lathe 9 inches in diameter. Find the breadth, approximately, of a single belt which will do to connect the two. Thickness as in Example 4.
7. A machine which requires 3 HP is driven from a shaft which revolves 200 times per minute. If the pulley on the shaft is 20 inches in diameter, find approximately the proper cross-section for belt.
8. A dynamo machine runs at 600 revolutions per minute, and gives out a current of 80 amperes, with an E.M.F. of 110 volts. If the efficiency of the dynamo is 80 per cent., find by the easy rule the proper breadth of single belt $\frac{1}{4}$ inch thick to use, the dynamo pulley being 10 inches in diameter.

9. It is required to transmit 6 HP from a pulley 4 feet in diameter to one 1 foot 6 inches in diameter by means of a belt, the centres of the pulleys being 12 feet apart. If the larger pulley revolves 180 times per minute, and the coefficient of friction between the belt and smaller pulley be ·35, find the angle of lapping on the smaller pulley, its speed, and the breadth of single belt $\frac{7}{32}$ inch thick which will be necessary.
10. Calculate the cross-sectional area of a belt which will give 10 HP to a pulley 14 inches in diameter, which revolves 150 times a minute, the angle of lapping being 120°, and $\mu = \cdot 4$.
11. It is necessary, in a certain machine, to transmit 3 HP to a pulley 16 inches in diameter, and revolving 100 times per minute, by means of a belt which laps only 60° on the pulley. Find what is the least pull in the slack side of the belt which will prevent slipping, the corresponding pull in the tight side, and the breadth of single belt $\frac{7}{32}$ inch thick which will drive safely. $\mu = 0 \cdot 4$.
12. A rope 6 inches in circumference is moving at 4,000 feet per minute. What HP will it transmit?

Rule for rope belts. $N - M = 7 \cdot 81c^2$, where c is the circumference of the belt in inches.

13. Rope belting is employed to drive machinery from the fly-wheel of a steam-engine the diameter of which is 8 feet. If the ropes employed are 5 inches in circumference, how many of them will be required to transmit 120 HP, the fly-wheel making 96 revolutions per minute?

ANSWERS.

1. 912 revolutions per minute.
2. 1·14 HP. 3. 316·1 lb.
4. $b = 7 \cdot 65$ inches. 5. $b = 6 \cdot 09$ inches.
6. $b = 1 \cdot 69$ inch. 7. ·573 square inch.
8. Breadth, 7·5 inches. (See Lesson VI., page 30, for measurement of HP electrically.)

9. $\theta = 2{\cdot}93$ radians. 480 revolutions per minute. Breadth, 1·89 inch.
10. 3·2 square inches.
11. Pull in slack side, 454·5 lb. Pull in tight side, 690·84 lb. Breadth of belt, 9·56 inches.
12. 34·08 HP. 13. Answer, 8·4; hence 9 ropes are required.

LESSON XIII.

LENGTHS OF BELTS.

For crossed belts the length L *is given by the rule*—

(1) $L = (D + d)\left(\frac{\pi}{2} + \theta\right) + 2c \cos \theta$, where θ may be found from the fact that $\sin \theta = \frac{D + d}{2c}$. (In finding $\frac{\pi}{2} + \theta$, it must be remembered that θ is expressed in radians. θ in degrees $\times$ ·0175 $= \theta$ in radians.) (Fig. 30.)

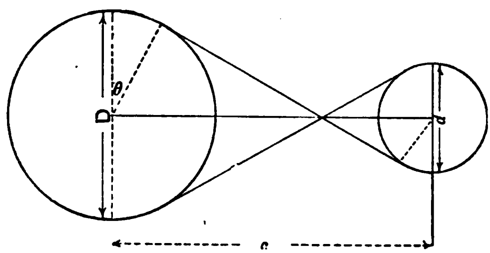


Fig. 30.

For open belts—

(2) $L = (D + d)\frac{\pi}{2} + 2c\left\{1 + \frac{(D - d)^2}{8c^2}\right\}$ is a very close approximation, and is the rule usually employed.

In designing stepped cones to work with a *crossed* belt, it is only necessary to have the *sum of the diameters of two corresponding steps a constant quantity* in order that the same belt may work on all the steps. For an open belt the calculation is more difficult.

Let S represent the sum of the diameters of two

corresponding steps, δ their difference, n the speed of one *driven* step, and let these quantities for successive pairs of steps, 1, 2, etc., be distinguished by the suffixes 1, 2,

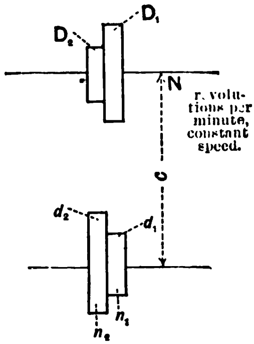


Fig. 31.

etc., as in the sketch (Fig. 31), then, since L is constant, it is easy to show that—

(3) $S_2 = S_1 + \frac{\delta_1^2 - \delta_2^2}{2\pi c}$. Combining this with the rule for speeds—

$$D_1 N = d_1 n_1$$
$$\text{and } D_2 N = d_2 n_2$$
$$\therefore \frac{D_2 - d_2}{D_2 + d_2} = \frac{n_2 - N}{n_2 + N}$$

(the quantities D_1, N, n_1, and n_2 are here supposed given),

(4) We get $D_2 - d_2 = \delta_2 = \left(\frac{n_2 - N}{n_2 + N}\right) S_2$, whence (3) becomes

(5) $S_2 = S_1 + \frac{\delta_1^2 - \left(\frac{n_2 - N}{n_2 + N}\right)^2 S_2^2}{2\pi c}$. This gives a quadratic for S_2, which, being found, can be used in (4) to find δ_2, and hence D_2 and d_2. The practical rule given by Unwin is—"Calculate the diameters D_2 and d_2 as if for a crossed belt, then taking δ_2 = the difference

of these diameters, find the value of S_2 by equation (3). From that value of S_2 re-calculate the diameters D_2 and d_2, using equation (4) and the rule for speeds." This process may be repeated till the answer is obtained with sufficient accuracy. The rules given are of course true for *any* two successive pairs of steps. It will easily be seen that $S_2 + \delta_2 = 2\ D_2$, and $S_2 - \delta_2 = 2\ d_2$, etc.

Example.—Two shafts are 10 feet apart. On one shaft, which rotates at the constant speed of 150 revolutions per minute, is a stepped cone, the largest diameter of which is 3 feet; this drives the smallest step of a similar pulley on the other shaft by means of an open belt. Find the diameters of steps which will cause the driven shaft to revolve at 200 and 60 revolutions per minute, and which will work with the same belt.

First find the diameters as for a crossed belt. Let the diameter of the given step be called D_1, the next diameter on the same pulley D_2, etc., corresponding steps on the driven shaft being denoted by d_1, d_2, etc.

From rule for speeds—

$$D_1 \times 150 = d_1 \times 200 \therefore \frac{D_1}{d_1} = \frac{200}{150} = \frac{4}{3}.$$

$$\text{But } D_1 = 3 \text{ feet} \therefore d_1 = \frac{3}{4} \times 3 = 2{\cdot}25 \text{ feet.}$$

Also (a)—

$$D_2 + d_2 = D_1 + d_1 = 5{\cdot}25$$

(using the rule for crossed belt).

$$\text{And } D_2 \times 150 = d_2 \times 60 \therefore \frac{D_2}{d_2} = \frac{60}{150} = \frac{2}{5}$$

$$\text{or } D_2 = d_2 \times \frac{2}{5}.$$

Substituting this value in equation (a), we find

$$d_2\left(1 + \frac{2}{5}\right) = 5{\cdot}25,$$

$$\text{or } d_2 = \frac{5{\cdot}25 \times 5}{7} = 3{\cdot}75 \text{ feet,}$$

$$\therefore D_2 = 5{\cdot}25 - 3{\cdot}75 = 1{\cdot}5 \text{ foot}$$

But these values are not correct for an *open* belt. We now proceed to correct them. Assume that $d_2 - D_2 = \delta_2 = -2{\cdot}25$ is not far wrong; use this value to find S_2 ($= D_2 + d_2$) from equation (3).

$$S_2 = 5{\cdot}25 + \frac{({\cdot}75)^2 - (2{\cdot}25)^2}{6{\cdot}2832 \times 10}$$
$$= 5{\cdot}25 + \frac{{\cdot}5525 - 5{\cdot}0625}{62{\cdot}832}$$
$$= 5{\cdot}25 - {\cdot}07178.$$
$$\therefore S_2 = 5{\cdot}1782.$$

Put this value into equation (4) to find a new value of δ_2. Thus—

$$\delta_2 = \left(\frac{60 - 150}{210}\right) 5{\cdot}1782.$$
$$\therefore \delta_2 = -\frac{3}{7} \times 5{\cdot}1782 = -2{\cdot}2192.$$

We have now obtained values of S_2 and δ_2 *corrected once.* To correct again, use the last value obtained for δ_2 in equation (3) to find S_2.

$$S_2 = 5{\cdot}25 + \frac{({\cdot}75)^2 - (2{\cdot}2192)^2}{62{\cdot}832} = 5{\cdot}25 - {\cdot}06959 = 5{\cdot}1804.$$
$$\therefore \delta_2 = -\frac{3}{7} \times 5{\cdot}1804 = -2{\cdot}2201.$$

Whence $S_2 + \delta_2 = 5{\cdot}1804 - 2{\cdot}2201 = 2{\cdot}9603 = 2\,D_2.$

$$\therefore D_2 = 1{\cdot}48 \text{ feet,}$$

and $2\,d_2 = S_2 - \delta_2 = 5{\cdot}1804 + 2{\cdot}2201 = 7{\cdot}4005,$
and $\therefore d_2 = 3{\cdot}7$ feet, corrected twice.

It may seem that these corrections are so small that they might be neglected; but probably these small corrections in diameter, if neglected, would prevent the belt from working well on any except one pair of the steps. The same method may be employed to find other diameters D_3 and d_3 from the now known values of D_2 and d_2.

Numerical Examples.

1. The centres of two pulleys, $3\frac{1}{2}$ and $1\frac{1}{2}$ feet in diameter respectively, are 10 feet apart. Find the length of crossed belt required.

2. Two pulleys whose diameters are 4 and 2 feet respectively, their centres being 8 feet apart, are connected by an open belt, find the length of belt necessary.
3. Suppose the coned pulleys (with 2 steps each) shown in Fig. 31 had the following data: $D_1 = 4$ feet, $d_1 = 1\frac{1}{2}$ foot, $N = 100$ revolutions per minute, $n_1 = 266 \cdot 6$, $n_2 = 80$, $c = 8$ feet. Find the remaining diameters, the belt being open.
4. The centres of two shafts are 8 feet apart. On these shafts, pulleys of 14 inches and 10 inches diameter respectively are keyed. Find the length of crossed belt required to connect the two pulleys. If the 14-inch pulley formed the smallest step, and the 10-inch pulley the largest step, of two coned pulleys respectively, and if the 14-inch pulley is on a shaft which rotates at the constant speed of 150 revolutions per minute, find the diameters of the remaining three steps of each cone, the required speeds of the driven shaft being 210, 300, 400, and 500 revolutions per minute.
5. A counter-shaft revolves at 150 revolutions per minute. On this shaft is a stepped cone, which drives another stepped cone on a machine by means of a crossed belt. If the diameter of the smallest step of the pulley on the counter-shaft is 6 inches, the diameter of the corresponding step on the machine being 15 inches, find the diameters of the remaining three steps of the two cones. The required speeds are 60, 120, 200, and 250 revolutions per minute.
6. A shaft revolves at the constant speed of 264 revolutions per minute. On this shaft is a stepped cone, with four steps, driving a similar cone on another shaft by means of an open belt. If the diameter of the largest step of the cone on the driving shaft is $23\frac{7}{8}$ inches, and the driven shaft is to go at 531, 324, 266, and 149 revolutions per minute respectively, find the diameters of all the remaining steps. The centres of the shafts are 9 feet $7\frac{3}{4}$ inches apart.

ANSWERS.

1. Length of belt, 28·48 feet. 2. Length of belt, 25·55 feet.
3. $D_2 = 2·496$ feet, $d_2 = 3·120$ feet.
4. Length of belt, 19·267 feet.

$D_2 = 16$	inches.	The diameters of the pulley on the shaft which rotates with constant speed are distinguished by capital letters.
$d_2 = 8$	"	
$D_3 = 17·45$	"	
$d_3 = 6·55$	"	
$D_4 = 18·46$	"	
$d_4 = 5·54$	"	

5. $D_2 = 9·335$ inches.
$d_2 = 11·66$ "
$D_3 = 12$ "
$d_3 = 9$ "
$D_4 = 13·125$ "
$d_4 = 7·875$ "

6. $d_1 = 11·86$ inches
$D_2 = 19·596$ "
$d_2 = 15·963$ "
$D_3 = 17·84$ "
$d_3 = 17·706$ "
$D_4 = 12·873$ "
$d_4 = 22·809$ "

LESSON XIV.

DYNAMOMETERS OR WORK-MEASURING MACHINES.

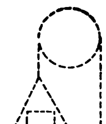
A "DYNAMOMETER," as the name implies, is a "measurer of force," and the older forms of this apparatus were somewhat similar to our spring-balance. Modern dynamometers are *measurers of energy*, and may be divided into two classes: *absorption* and *transmission* dynamometers; the former wasting the energy by friction whilst measuring it, the latter transmitting it without much waste. If a record of the time, during which a given amount of energy is wasted or transmitted, be kept, the instrument may be used to measure power, and it is to this purpose that most modern dynamometers are applied.

ABSORPTION DYNAMOMETERS.

The following rule will give the power *absorbed* by any absorption dynamometer in which the energy is wasted by the friction of a brake-block or strap on a pulley, viz.:—The algebraic sum of the moments of all the externally applied forces, taken about the centre of

the brake-pulley and measured in pound-feet, multiplied by the angular velocity of that pulley in radians per minute, and divided by 33,000, gives the HP.

The Prony Brake.—One of the best-known and most widely used of absorption dynamometers is the Prony brake. It was invented about the year 1820 by Piobert and Fardy, but improved and brought into successful use by Prony. There are many modifications of it in use, but the form shown in Fig. 32 will best illustrate the characteristic features of the instrument. It con-



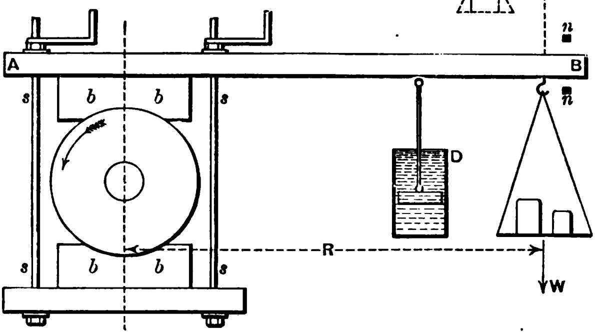


Fig. 32.

sists of a beam A B to which one of the brake-blocks *b b* is fastened, the other being fastened to a similar but shorter beam, and both pressed against the rim of the brake-pulley (which should be strong and truly cylindrical, and which rotates as shown by the arrow, being driven by the motor to be tested) by the screws *s s*, which can be tightened during the test till the beam just floats in the horizontal position and midway between the two stops *n n*. In this position the moment of the total friction between the brake-blocks and pulley is just balanced by the moment of the weight W, the beam itself being counterpoised by a weight shown in dotted lines. The moment of the weight W is W × R, hence if W is in

pounds and R in feet, the HP absorbed is given by the rule,

$$HP = \frac{W \times R \times 2\pi n}{33000},$$

where n is the number of revolutions the brake-pulley makes per minute. The dash-pot D is attached to a wall or some separate support, and is filled with oil or other fluid, its object being to still the vibrations of the beam A B.

Carpentier's Dynamometer.—Ingenious methods have been adopted in order to make a dynamometer automatically adjust itself to variations in the coefficient of friction between the rubbing surfaces. Among these, that adopted by M. Carpentier (Fig. 33) is noteworthy. The shaft s, conveying the power to be measured, carries two pulleys, A being fast and B loose on the shaft. The pulley B has a flange F, in which is fastened the centre of a rope which is coiled round the two pulleys in opposite directions as shown, and which bears two unequal weights W and w. The direction of rotation is such as to lift the larger weight if the rope does not slip on the pulley A. If an accidental increase of friction *does* take place, the weight W is lifted, and the amount of lapping of the belt on A, and hence the total friction is diminished; so that there is always just sufficient rope on A to cause a steady waste of all the energy supplied. The weights W and w are carefully adjusted, so that the automatic adjustment will only have to compensate for *small* changes of friction or of energy supplied.

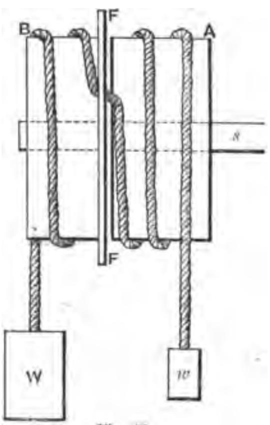


Fig. 33.

Raffard's Dynamometer is on the same principle—is, indeed, a modification of Carpentier's. In it the larger

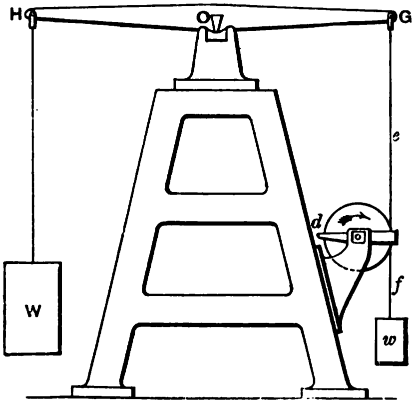


Fig. 34.

weight, instead of being fastened directly to the belt or rope, is hung from one end of a lever with equal arms, the belt being fastened to the other end. This arrangement will readily be understood from an inspection of Figs. 34

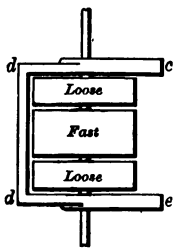


Fig. 35.

and 35, which are respectively an elevation of the apparatus and a plan of the pulleys. The weight *w* (Fig. 34) hangs from the belt *f*, which takes one half turn

over the top of the fast pulley, and is fixed to the cross-bar *d d* (Fig. 35); two other belts *e* are fixed to *d*, lapped over the lower half of each loose pulley, and are then attached to the end of the lever G H (Fig. 34), from the other end of which the larger weight W is suspended. The motor to be tested is coupled directly to the shaft, which carries these pulleys, by means of

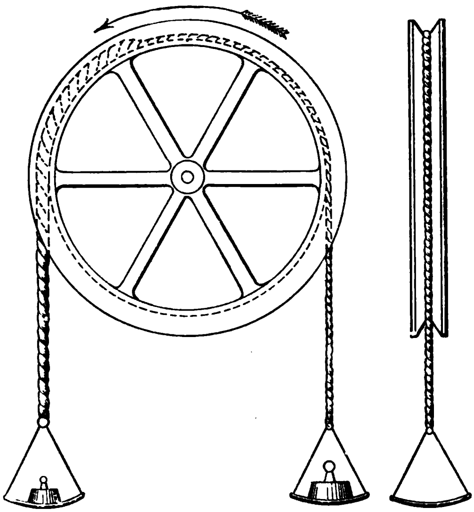


Fig. 36.

a universal joint. The rule for calculating HP, in this and the last apparatus, is the same as the rule already given for the Prony brake, except that the *net* load W − *w* is substituted for W, and R is the radius of the pulleys.

Ayrton and Perry's Absorption Dynamometer.—A much simpler and equally effective method of adjustment has been devised by Professors Ayrton and Perry. It is shown in Fig. 36. The belt or rope is of unequal roughness—shown as unequal thickness in the figure—

and as the coefficient of friction diminishes by the wearing of the surfaces of the belt and pulley, a rougher portion of the belt is drawn on to the pulley and the friction again automatically increased. The pulley is generally a flat one with projecting flanges, and the belt employed an ordinary leather or cotton belt, the necessary roughness being obtained by lacing the belt with a rough thong or lace. The rule for HP is the same as in the last case.

The Appold Brake.—This dynamometer of Messrs. Amos and Appold, which was formerly used in testing

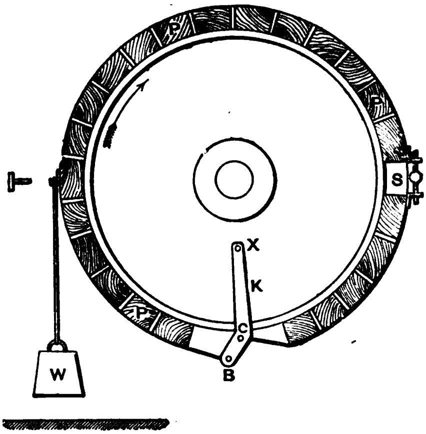


Fig. 37.

engines at the Royal Agricultural Society's shows, has attracted considerable attention.

Its very ingenious automatic adjustment will be understood from an examination of Fig. 37.

It will be seen that there is a screw S at one side for adjusting the brake-strap P to the proper tightness. The *automatic* adjustment is by means of the bent lever K, which, if the friction becomes a little too great, moves round a little in the direction of the arrow-head about the pivot X, the weight W being lifted. The point B, being

farther from the pivot than C, moves farther, hence the effect is the same as if the brake-band had been *lengthened*;

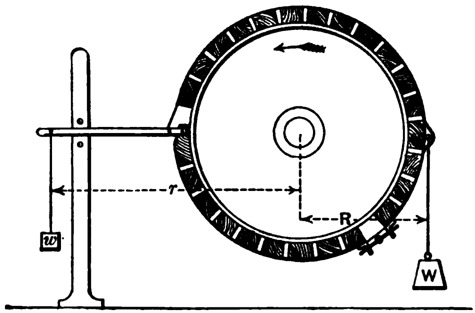


Fig. 88.

the friction between it and the pulley being correspondingly diminished. There is, therefore, always a tendency to return to a condition of equilibrium; for should the

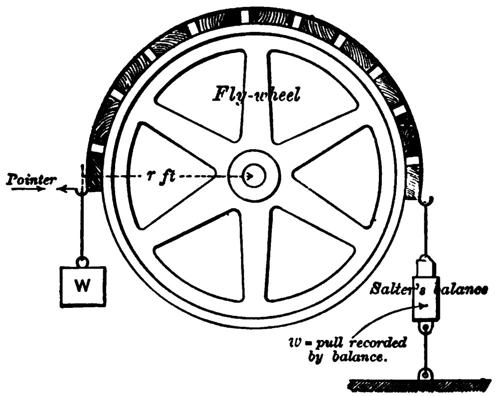


Fig. 89.

friction become *less*, the opposite effect is produced. This is a very ingenious arrangement, but it has been pointed out that there is an *unmeasured* force at the point X,

which is greater the more the compensating lever comes into play; hence the indications cannot be relied on for accuracy. If, however, the compensating lever is arranged as shown in Fig. 38, *all* the forces can be taken into account. If things are so adjusted that the lever floats midway between the stops, the power absorbed is given thus:—

$$HP = \frac{(W \times R - wr)\, 2\pi n}{33000}.$$

In recent tests of engines, conducted with great care, simpler forms of dynamometers, such as those shown in Figs. 39 and 40,* were used. The reader will have no difficulty in understanding the action of each from an inspection of the figure. The brake-strap or rope may lap either half or completely round the pulley to which the power to be measured is supplied. The rule for both of these cases is

$$HP = \frac{(W - w)\, r \times 2\pi n}{33000},$$

W and w being measured in pounds, r in feet, and n in revolutions per minute. These brakes are easy to construct, work well, and give accurate results. The foregoing dynamometers are usually employed to measure the power given out by prime movers and motors, such as steam- or gas-engines, electro-motors, etc. On the other hand,

TRANSMISSION DYNAMOMETERS

are generally used to measure the power given to *power-absorbing* machines, such as dynamos, pumps, etc. Some of them can remain in position permanently, so as to show at any time the amount of power passing through them to one or a number of machines.

Ayrton and Perry's Dynamometer Coupling.—Such an apparatus is the dynamometer coupling of Professors

* Figs. 39 and 40 are, by kind permission, copied from the report on Mr. W. W. Beaumont's paper on "Friction-Brake Dynamometers" in the Minutes of Proceedings of the Institution of Civil Engineers, vol. xcv.

Ayrton and Perry. As shown in Fig. 41, it is really a coupling for connecting two lengths of shafting—not rigidly, as is usually the case, but through the medium of spiral springs, which yield and allow a certain amount

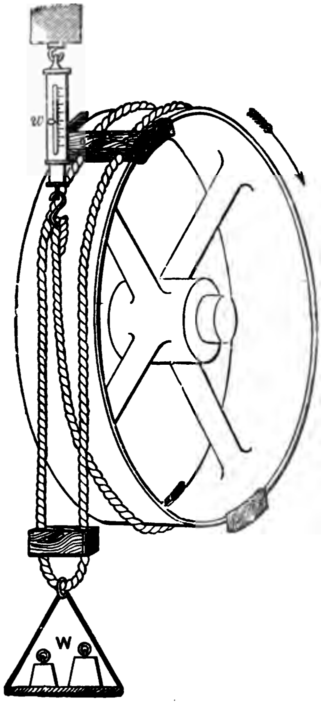


Fig. 40

of angular motion of the one half of the coupling or one length of shaft, relatively to the other. This yielding or relative motion is magnified in a most ingenious way, which, however, is somewhat difficult to understand from the diagram. The bar D, instead of being as shown, is really fastened to the *further* half of the coupling, and a

pin from it engages a little link that acts on the light pointer E pivoted at F. The direction of rotation is that shown by the arrow, and the further length of shaft tends to get in advance of the nearer when power is being transmitted; hence the bar D pulls the end B of the pointer in nearer the centre of the shaft as the amount of power transmitted becomes greater. The pointer at B carries a bright silvered bead, which, rotating near the blackened disc A, seems to describe bright circles; and a scale arranged in front of the pointer gives the radius of that circle — or, rather, shows the distance, radially, of the circle the bead describes from that described by it when no power is transmitted. The reading on the scale, then, is a measure of the HP (really the *torque* or turning moment) transmitted if the speed of the shaft is known and constant.

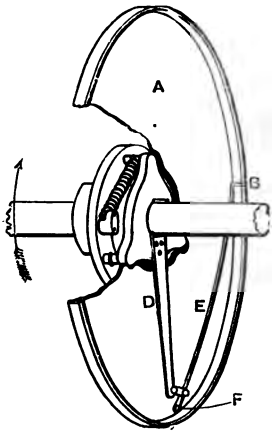


Fig. 41.

Smith's Transmission Dynamometer or *Ergometer* is a very useful and ingenious instrument. It is shown in

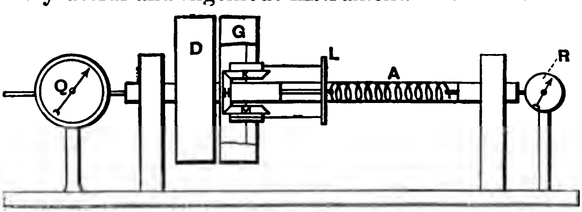


Fig. 42.

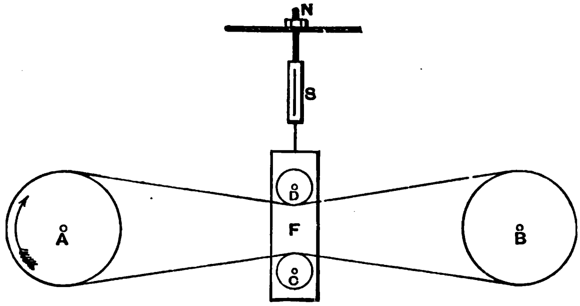
section in Fig. 42. It consists of a hollow shaft on which are mounted two pulleys, D fast and G loose on

the shaft. The pulley D has attached to it the bearings for a bevel-wheel H, which gears with two others, K and M, the bearings of which are fixed to G. Wheels K and M have each a little drum attached, on which steel wire is wound in opposite directions in the two cases, the wire being attached to a cross-bar L passing through a slot in the shaft, this cross-bar being fastened to the end of a spiral spring A occupying the centre of the hollow shaft. Pulley D receives the power by a belt, and G gives it off after measurement. When power is transmitted, the pulley G tends to lag behind D, and this relative motion of the pulleys causes a relative motion of the bevel-wheels; K and M moving round on H and winding up the steel wire, therefore elongating the spring. This elongation is really a measure of the *torque* applied to the shaft, and is shown by the pointer on the dial Q, the speed of the shaft being given by the speed-indicator R. Hence the two factors of HP, *torque* and *speed*, are given, and the HP is obtained by multiplying the two readings together and dividing or multiplying by a constant number, which has been determined for the particular apparatus. This dynamometer has the great advantage over some others of giving the HP at *any* speed without the necessity of introducing a correction.

Hefner-Alteneck's Belt-tension Dynamometer.—This dynamometer was first brought to public notice in Great Britain in 1879. It is designed to measure the *difference of the pulls* in the two sides of a belt which transmits power. The principle of the instrument will be understood from Fig. 43, which, however, is only diagrammatic.

The pulley A receives the power and drives the pulley B by means of a belt, which is deflected over the guide-pulleys C and D, mounted on a frame which is guided to move vertically. The lower half of the belt being the tighter portion—since A drives in the direction shown—there is a tendency to move the frame F, carrying the guide-pulleys, downwards against the pull of the spiral spring S, the nut N of which is tightened till the pulleys assume a symmetrical position, as in the figure. When a larger amount of power is transmitted, it

takes a greater pull in the spring-balance to keep the guide-pulleys in the proper position; in fact, the pull recorded in the spring-balance is a measure of the HP transmitted at any constant speed; in reality it measures the difference of pull in the two sides of the belt, which the reader is already aware is proportional to the HP transmitted, if the speed does not vary. The actual form of the instrument is different from that shown, but the figure illustrates the principle on which it acts. An instrument very similar to that shown in



Fig, 43.

Fig. 43 has been used with success by Professor Elihu Thomson, of the United States, in measuring the power given to dynamo machines. In that case Professor Thomson suspended the frame F from the short arm of a lever the longer arm of which carried a movable weight. The lever was graduated so that the position of the movable weight indicated the HP transmitted at a certain speed.

Froude's or Thornycroft's Dynamometer.—This belt-tension dynamometer is shown diagrammatically in Fig. 44. The balancing pull is here applied to the belt, *not* at right angles to its length, as in the last dynamometer described, but parallel to it. The driven pulley D rotates, as shown by the arrow, hence the upper is the tighter side of the belt. Let N be the tension in the tight, and M that in the slack side, both in pounds; then two pulls N in

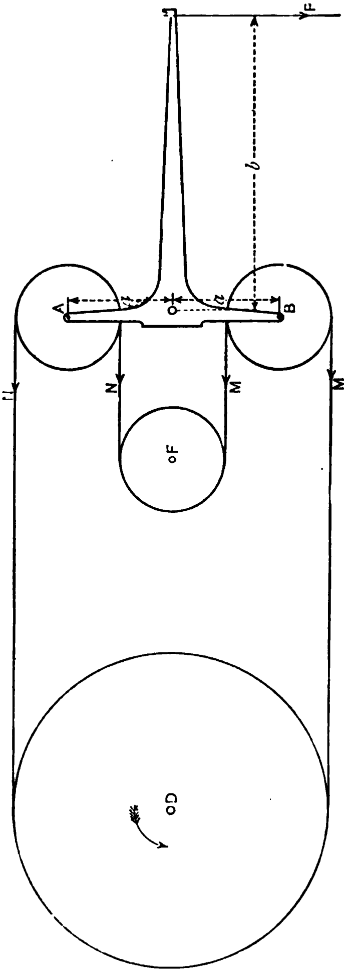


Fig. 44.

the belt on different sides of the small pulley A are nearly equivalent to the pull 2 N at the centre of the

pulley. Hence from the equilibrium of the forces shown, P being the adjustable balancing force;

$$P \times b = 2\,N \times a - 2\,M \times a.$$
$$= 2\,a\,(N - M)$$
$$\text{or } N - M = \frac{P \times b}{2\,a};$$

and as before

$$\frac{(N - M)\text{ vel. of belt in feet per minute}}{33{,}000} = \text{HP transmitted.}$$

In the above calculation friction is neglected. The pulls are *not* quite equal on both sides of each pulley, but the result is nearly right, and friction can be obtained only by experiment. Many modifications of this dynamometer have been used both in Great Britain and in America.

The student will gain much useful information by studying the way in which the elementary laws of mechanics are applied in the various machines described.*

Numerical Examples.

1. A Prony brake is used to measure the useful HP of a certain steam-engine, the testing pulley being keyed on the crank-shaft of the engine. It is found that there is equilibrium when a weight of 102 lb. is suspended from the lever, the centre of gravity of this weight being 8 feet from the centre of the pulley. If the speed of the shaft is 98 revolutions per minute, find the useful HP.
2. A Raffard (Carpentier) dynamometer is used to measure the power given out by an electro-motor. If the weights W and w (Fig. 34) are 68 and 20 lb. respectively, the diameter of the brake-pulleys being 14 inches, and their speed 1,200 revolutions per

* For a fuller treatment of this subject the student is referred to the "Proceedings of the Institution of Civil Engineers for 1889," the *Electrician* for 1883-84, and the *Mechanical World* for 1889.

minute, find the useful HP of the motor. If the motor receives a current of 62 amperes at a pressure or electromotive force of 100 volts, find its efficiency.

Referring to Fig. 34 it will be seen that the lever G H has *equal* arms, hence it serves merely to transfer the force due to the weight W to the belts at *e*, the dimensions of this lever not entering into the calculation.

The dynamometer really consists of a fast pulley with a belt lapped half round it, one end being pulled by a force of 68 lb., and the other by a force of 20 lb.

The efficiency of the motor is the ratio of the power it gives out to that which it receives.

3. The HP of an engine is measured by a simple dynamometer, such as that shown in Fig. 36, the weights being 210 and 30 lb. respectively, and the speed of the pulley 150 revolutions per minute. If the mean radius of the circle described by the centre of the rope or belt is $2\frac{1}{2}$ feet, find the useful power of the engine.
4. In using the modification of the Appold brake, shown in Fig. 38, it was found that there was equilibrium when the larger weight was 118 lb., its distance from the centre of the shaft being 2 feet 3 inches; the smaller weight being 20 lb., and its distance 4 feet 8 inches; and the speed of the brake-pulley 216 revolutions per minute. Find the power absorbed.
5. In testing the efficiency of a certain machine, the power it received was transmitted through a Smith's transmission dynamometer; and it was noted that the dynamometer showed a torque of 480 pound-feet, and a mean speed of 178 revolutions per minute. Find the power transmitted.
6. By means of a Hefner-Alteneck's belt-tension dynamometer, it was found that the difference between the pulls in the two sides of a belt which transmitted power was 188 lb., the diameter of the pulley driven by the belt being 15 inches, and its speed 280 revolutions per minute. Find the amount of power transmitted by the belt.

7. In the dynamometer used by Prof. Elihu Thomson, as described at page 75, if the short arm of the lever carrying the movable weight was 1 foot, the weight of the lever itself being counterpoised; and if the angle between the two portions of the belt at each carrying pulley is 120°, find—neglecting friction—the distance apart of the graduations on the lever, each of which represents 1 HP at a belt-speed of 2,500 feet per minute, the movable weight being 20 lb.
8. In a dynamometer similar to that shown in Fig. 44, b was 32 inches, a 16 inches, P 40 lb. The speed of the driving pulley D was 200 revolutions per minute, and its diameter 4 feet. Find the HP transmitted.

ANSWERS.

1. 15·23. 2. 6·4 HP, nearly. Efficiency, 77 per cent.
3. 12·8. 4. 7·08 HP. 5. 16·26 HP.
6. 6·26 HP. 7. Distance, ·66 foot. 8. 3·04 HP.

LESSON XV.

STRENGTH OF MATERIALS.

"STRESS" is the mutual action believed to take place between two bodies in contact, or between the particles of a body subjected to load. Stress is generally measured in terms of applied force, and expressed as the *force per unit area* of cross-section, in such a case as that of a piece of material subjected to pull. The stress in this case is *tensile* stress.

A short pillar supporting a load is subjected to stress of a similar kind if the load acts perfectly in the axis of the pillar; the stress in this case being termed *compressive* stress.

A third kind of stress is that due to a *tangential* force. The particles in this case are subjected to a mutual sliding action; this is called *shear* stress, and the resulting action is termed *shearing*. A plate of iron whilst being shorn in the shearing machine is subjected to this stress.

Another kind of stress is that experienced by a body which is subjected to a uniform pressure, of the nature of fluid pressure, all over its surface. A small block of material immersed in the water in an hydraulic press has this kind of stress. It may be called *hydrostatic*, or volumetric, stress.

Hooke, about the year 1676, first enunciated his celebrated law, deduced from certain experimental observations—the law being that, within the limits of stress for which the material is perfectly elastic,—

Stress is proportional to Strain.

For tensile or compressive stress and strain, this law becomes

$$\text{Stress} = \text{E} \times \text{strain},$$

where E is a coefficient called "Young's Modulus of Elasticity."

For shear stress and strain, the law is—

$$\text{Stress} = \text{N} \times \text{strain},$$

where N is the "modulus of rigidity," or modulus of torsional elasticity of the stuff.

For the remaining kind of stress and strain referred to, the law is—

$$\text{Stress} = \text{K} \times \text{strain},$$

where K is the "modulus of cubic compressibility."

All these moduli are generally, in Britain, expressed in lb. per square inch; in other words, the modulus is really that stress which would produce unit strain if the stuff remained elastic. For instance, in the case of a tie-rod the modulus E is that tensile stress which would make the elongation *equal* to the original length if we can

imagine any material as being elastic for such a high stress. Of the three moduli, that of most importance in engineering calculations is Young's modulus. The following table gives average values of the moduli for a few materials:—

MODULI OF ELASTICITY.

Material.	E.	N.	K.
Iron, cast... ...	17,000,000	6,300,000	14,000,000
" wrought	27,000,000 to 29,000,000	10,500,000	20,000,000
Steel, mild ...	30,000,000	11,000,000	—
" tempered	33,000,000 to 36,000,000	13,000,000 to 14,000,000	26,000,000
Brass	12,000,000	3,500,000	—
Copper	15,000,000	5,600,000	24,000,000
Phosphor bronze	13,500,000	5,400,000	—
Aluminium "	14,800,000	—	—
Gun metal ...	11,500,000	—	—
Wood, yellow pine	1,400,000	90,000	—
" pitch pine	2,000,000	—	—
" oak ...	1,800,000	82,000	—
Water	—	—	300,000

ULTIMATE AND PROOF STRESSES.

The *ultimate stress*, or strength, of any material is the intensity of stress, of any given kind, required to produce fracture. It is estimated as if the section of the material remained of the same size up to breaking, which is not the case, as the section, in the case of tensile stress for instance, usually diminishes before fracture takes place.

The *proof stress*, or elastic strength, of any material is the greatest stress the material will bear repeatedly without taking a "permanent set," or without permanently changing in shape.

The *working stress* is that stress which is considered permissible in practice, and is usually found by dividing

the ultimate or breaking stress by a number called the *factor of safety.*

The student may wonder why engineers do not take a certain fraction of the elastic stress as the safe working stress. Under the heading "The Elastic Limit" the editor of *Engineering** some time ago gave an excellent answer to this question. He showed that the method has been, to a certain extent, discredited by experimental facts.

Wöhler, in and since 1871, made experiments on the effect of "live" loads. He put on and took off successively certain loads for a great number of times, and it was found that in every case the *breaking* stress was much below that for a steady quiescent load, and that, in fact, if one designed a piece from the elastic stress for quiescent loads, it is possible the piece might *break* much below that limit if the load changed its nature. The elastic limit is really a *variable* thing depending on the nature of the load.

Another reason is, there are *several* elastic limits for the same material. Thus there is, first, the maker's elastic limit for the material as it comes from the rolls; second, the real elastic limit of the stuff after being brought to a "state of ease" by taking out the initial strains; and, third, the elastic limit when the piece is loaded in various ways, which will depend on the load.

Professor Bauschinger clearly pointed out that the *elastic limit changes with the previous history of the specimen*, and he made some most important experiments to find whether raising the elastic limit *in tension*, by subjecting the piece to loads slightly exceeding that limit, had any effect on the elastic limit for compression. He found that raising the former depressed the latter, and if the stress were repeatedly alternated the elastic limit took up a position about midway between the two limits, but at a *much lower* point than for quiescent loads. Straining a bar slightly beyond one elastic limit does not seem to hurt it much; it is straining it beyond its *two* elastic limits which produces the evil result. If the true elastic stress for the kind of stress to which a bar is going to be subjected were known, no doubt the "elastic limit" method would be the proper one to adopt; but in the present state of our knowledge engineers are not far wrong to adhere to the older and more certain method.

Our knowledge of the ultimate and elastic stresses which will be borne by materials such as iron and steel is largely due to experiments which have been made by the aid of testing machines such as that shown in Fig. 45, which is a "Wicksteed single lever" machine capable of

* *Engineering*, August 7th, 1891.

applying a pull of 100 tons to the specimen. The specimen *a* is connected by proper gripping arrangements, *g g*, at one end to the short arm of a strong lever L, the fulcrum of which is nearly over the centre of the strong central support, and at the other to a powerful cross-head connected with the ram of an hydraulic press. It may be said, therefore, that the pull on the specimen is produced by the pumps which work the press, and balanced by the pull due to the movable weight W. This weight can be moved along the lever by a screw and worm-wheel (not shown), the screw being worked by the spur-wheels *m n* driven through bevel wheels *b*, either by steam-power from the pulley P, or by the hand-wheel H. The zero position of the movable weight is shown by the shaded disc, the lever L balancing about the fulcrum when the weight is in this position. The standard R prevents too great a descent of that end of the lever when the specimen breaks. The machine can be arranged for compression, torsion, and bending tests, in addition to the tension adjustment shown in the figure. I am indebted to the makers, Messrs. Buckton & Co., of Leeds, for the view shown in Fig. 45.

Space does not permit a description of other testing machines which have been employed for important tests. Many of these differ only in detail from that described. In small machines the hydraulic press is often replaced by a screw worked by a worm and worm-wheel which is actuated either by hand or power. In other machines, again, the "diaphragm" principle is adopted, *i.e.*, the pull is applied to the specimen through a cross-head connected with the flexible end of a short wide cylinder, containing a fluid such as water, the pressure of this fluid affording a ready means of indicating the load or stress. The student is referred to the "Minutes of Proceedings of the Institution of Civil Engineers" for further information on this subject.

The following table of stresses has been compiled from average values stated by various authorities—mainly those given in the newer editions of some of Professor Unwin's works:—

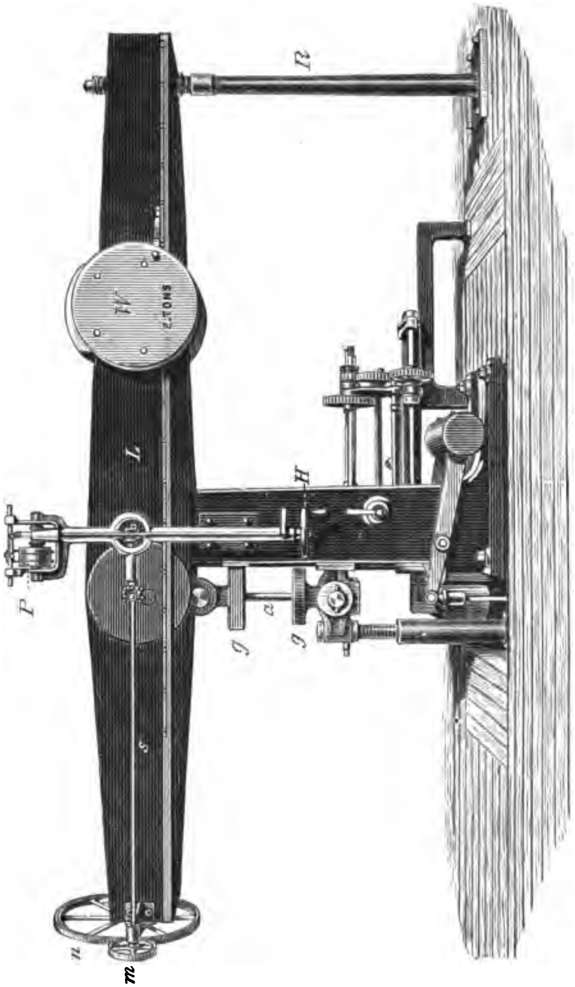


Fig. 45.

ULTIMATE AND ELASTIC STRESSES.* (*Pounds per square inch.*)

	Breaking Stress.			Elastic Stress.		
	Tension.	Compression.	Shearing	Tension.	Compression.	Shearing.
Wrought-iron bars .	53,000	50,000	40,000	30,000	30,000	22,000
„ „ plates	48,000	—	36,000	24,000	24,000	15,000
„ „ wire .	78,000	—	—	—	—	—
Steel (mild) plates .	70,000	—	54,000	42,000	—	43,500
„ cast, untmprd.	118,000	—	—	80,000	80,000	64,000
„ „ tempered	—	—	—	190,000	190,000	145,000
„ „ piano-wire	300,000	—	—	—	—	—
Cast-iron … …	19,500	94,000	10,500	†10,500	21,000	8,000
Copper, cast … …	21,000	58,000	—	—	—	—
„ plates …	31,000	—	—	5,600	4,000	3,000
„ wire … …	51,500	—	—	—	—	—
Brass … … …	23,000	—	—	—	—	—
„ wire … …	50,000	—	—	—	—	—
Phosphor bronze …	58,000	—	43,000	19,500	—	14,500
„ „ wire	74,000 to 150,000	—	—	—	—	—
Gun metal … …	34,000	—	—	6,200	—	4,200
Delta metal, cast …	36,000	—	—	17,000	—	—
„ „ rolled .	74,000	—	—	51,000	—	—
Wood, along the grain						
Pine from …	13,000	7,000	650	—	—	—
Pine to … …	6,500	3,500				
Oak … … …	15,000	10,000	2,300	—	—	—

FACTORS OF SAFETY.

Material.	Dead, or steady, constant load.	Live or varying load.	Structures subjected to shocks.
Wrought-iron and steel	3 to 4	5 to 8 ‡	10 to 12
Cast-iron … … …	4 to 6	6 to 10	15
Timber … … … …	7	10 to 15	20
Brick-work and masonry	20	30	—

* These values are averages obtained, in some cases, between rather wide limits. The safe working stress can be obtained by dividing the ultimate stress by the proper factor of safety. For more detailed information, consult Professor Unwin's "Machine Design."

† It is doubtful whether cast-iron has any proper elastic limit.

‡ The higher number refers to alternating stresses.

TENSILE AND COMPRESSIVE STRESS.

$$\text{Stress} = \frac{\text{Total load in lb.}}{\text{Area of right cross-section in square inches}}.$$

Example.—A wrought-iron bar used as a tie-rod has to withstand a pull of 20 tons; find the proper area of its cross-section from the limits of stress given in the table on page 85, and using 5 as a factor of safety. If the section of the tie-rod is circular, find the diameter of the circle.

Referring to the table, the ultimate stress for wrought-iron bars is 53,000 lb. per square inch; hence, the safe stress is $\frac{53,000}{5}$, or 10,600 lb. per square inch.

$$\frac{\text{Total load}}{\text{Area of section}} = \text{stress} = 10,600,$$

$$\text{Or, } \frac{20 \times 2,240}{10,600} = \text{area required.}$$

Hence, the required area is 4·2 square inches.

NOTE.—The following is a convenient rule for finding the diameter of a circle of which the *area* A is given:—

$$d = 1{\cdot}128\ \sqrt{A} = 1\tfrac{1}{8}\ \sqrt{A}, \text{ nearly};$$

in this case $d = 1{\cdot}128\ \sqrt{4{\cdot}2} = 2{\cdot}312$ inches.

Numerical Examples.

1. A weight of 60 lb. is suspended by a wrought-iron wire; find the diameter of the thinnest wire which can be used with safety, the factor of safety being as in the last example. Breaking stress, 78,000 lb. per sq. inch.
2. The total force on a chisel is 56 lb., the area of the edge $\frac{1}{900}$ of a square inch. Find the pressure at the edge in lb. per square inch.
3. A wrought-iron tie-rod has to withstand a pull of 20 tons. If the safe stress of the material is 10,400 lb. per square inch, find the area of the cross-section. If the cross-section is circular, find its diameter.

4. The total load on a short cast-iron strut is 10 tons. Find the safe area of its cross-section. Safe stress, 15,500 lb. per square inch.
5. A wrought-iron pipe, used as a tie-rod, has to withstand a pull of 15 tons. If its internal is $\frac{5}{6}$ of its external diameter, find the proper external diameter. Safe stress as in Example 3.
6. A weight of 56 lb. is suspended by a wrought-iron wire. Find the diameter of the thinnest wire that can be used with safety. Factor of safety, 5.
7. A weight of 25·5 lb. is suspended by a wrought-iron wire, ·082 inch in diameter. Find the stress in the wire. Find also the greatest weight that the wire will support with safety. Factor as in last example.
8. What must be the height of a column of cast-iron which will, by its weight, produce that pressure at its base which would crush a small column of the same material? Crushing stress of cast-iron, 94,000 lb. per square inch.
9. A wall of brickwork, 3 feet thick, is supported on sandstone columns, 9 inches in diameter and 10 feet apart from centre to centre. To what height can the wall be safely carried, given that the crushing stress of sandstone is 4,000 lb. per square inch, and using 10 as a factor of safety?
10. If the columns in the last Example were of granite, the crushing stress of which is 7,500 lb. per square inch, find the height to which the wall can be carried.

STRENGTH OF CHAINS.

The following rule for the strength of small chains is given by Professor H. S. Hele-Shaw: $P = C A$, where P is the working pull in lb., A the area of a cross-section of the material in square inches, and C a constant = 10,000 for "single-jack" chain, 20,000 for "double-jack" chain, 50,000 for welded chain, and 100,000 for American "triumph non-welded" chain.

Professor Unwin gives $P = 11{,}200\ d^2$ as a safe working load for large iron chains, d being the diameter

of the iron in inches. The weight of such chain in lb. per foot varies from 9 d^2 to $9\frac{2}{3}$ d^2.

11. Find the working load for each of the kinds of chain mentioned, the diameter of the iron being ·18 inch.
12. Find the working load for an iron chain, stuff one inch diameter. Also for a chain the iron of which is $1\frac{3}{8}$ inch diameter. Find the weight of the latter chain per foot.

ANSWERS.

1. Diameter, ·069 inch. 2. 50,400 lb. per square inch.
3. Area, 4·3 square inches. Diameter, 2·34 inches.
4. 1·445 square inch. 5. External diameter, 3·66 inches.
6. Diameter = ·0676 inch.
7. 4,828·6 lb. per square inch. Greatest safe load, 82·38 lb.
8. 29,000 feet, nearly.
9. Greatest safe height of wall, 7·57 feet. 10. 14·19 feet.
11. 254·47, 508·94, 1,272·35, and 2,544·7 lb. respectively.
12. 11,200 and 21,175 lb. About 17 lb. per foot.

LESSON XVI.

STRESS AND STRAIN.

Stress = E × strain.

Also strain × length = elongation or shortening, according as the stress is tensile or compressive. The lengthening will be in the same units as the multiplier "length."

Numerical Examples.

1. A wrought-iron tie-rod, $3\frac{1}{2}$ square inches in cross-section, is subjected to a pull of 20 tons. Find the strain (given E = 29,000,000). If this tie-rod is 20 feet long, how much will it lengthen?
2. Find the elongation of the wire in Example 6, Lesson XV., if it is 20 feet long. E as in last example.
3. Find the amount by which the wire in Example 7 of Lesson XV., if 20 feet long, will lengthen with its load of 25·5 lb.

4. Determine the elongation of a steel bar, 2 inches square and 40 feet long, when subjected to a pull of 40 tons ($E = 30,000,000$).
5. The diameter of the piston of a steam-engine is 12 inches, the diameter of its rod $1\frac{3}{4}$ inch, the length of the crank 12 inches, the length of the steel connecting-rod 5 feet, and its diameter 2 inches. If the pressure of steam in the space round the piston-rod just before motion is 80 lb. per square inch, find the lengthening of the connecting-rod, the crank being midway between the dead points. If the piston-rod is 5 feet long, how much will it lengthen?

ANSWERS.

1. Strain = ·00044. If 20 feet long, lengthening will be ·105 inch.
2. Elongation = ·128 inch. 3. Lengthening, ·039 inch.
4. Elongation, ·358 inch.
5. Pull in connecting-rod, 9,038 lb. Lengthening of connecting-rod, ·0057 inch. Lengthening of piston-rod, ·00736 inch.

LESSON XVII.

ELASTICITY OF BULK. SHEARING. CONNECTION BETWEEN MODULI, ETC.

Change of hydrostatic pressure all over a body's surface in lb. per square inch = K × cubical strain.

$$\text{Cubical strain} = \frac{\text{change in volume}}{\text{original volume}}.$$

Shear stress = N × shear strain.

$$\text{Shear strain} = \frac{\text{distance moved parallel to fixed plane}}{\text{distance from that plane}}.$$

$$a = \frac{1}{E}. \qquad K = \frac{1}{3\,(a - 2b)}. \qquad N = \frac{1}{2\,(a + b)}.$$

Where { K is the modulus of elasticity of bulk.
N ,, ,, rigidity of the material.
(Both in lb. per square inch.)
b is the lateral strain, corresponding to a longitudinal strain a, or a longitudinal stress of 1 lb. per square inch.

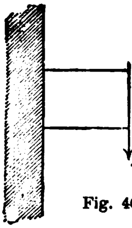
Numerical Examples.

1. Young's modulus of elasticity for wrought-iron is 29,000,000. K is 20,000,000. Find b.
2. From the now known values of a and b find N for wrought-iron.
3. Find the new volume of one cubic foot of water after being subjected to a pressure of 3 tons per square inch in an hydraulic press. K = 300,000.
4. A cube of wrought-iron, 2 inches in edge, is subjected to a hydrostatic pressure of 5 tons per square inch. Find its new volume. K = 20,000,000.
5. A sphere of copper, of 12 inches' diameter, is subjected to a hydrostatic pressure whose total amount is 360 tons. Find its new volume. K = 24,000,000.
6. A pine beam, 3 inches broad and 11 inches deep, projects one foot from a wall in which it is firmly fixed, and bears a load of 2 tons at its outer end. Find the deflection of that end due to shearing alone. N = 90,000.
7. The two halves of a flange coupling, which has to transmit 20 HP, are fastened together by 4 round wrought-iron bolts. If the centre of each bolt is 5 inches from the centre of the shaft, and if the shaft makes 120 revolutions per minute, find the proper diameter for each bolt. Safe shear stress of wrought-iron here taken as 7,800 lb. per square inch.

ANSWERS.

1. $b = \dfrac{1}{112{,}258{,}065}$.   2. N = 11,523,178.
3. New volume = ·9776 cubic foot.
4. New volume, 7·99552 cubic inches.
5. Old volume, 904·78 cubic inches. New volume, 904·7128 cubic inches. The area of the surface of a sphere of radius r is $4\,\pi\,r^2$. The volume of a sphere $= \dfrac{4}{3}\,\pi\,r^3$.

6. Deflection due to shearing is ·018 inch. The following is an explanation of the method of working such Examples (Fig. 46):—



2 tons

Fig. 46.

Area of end of beam = 33 square inches.

$$\text{Shear stress} = \frac{2 \times 2{,}240}{33} \text{ lb. per square inch.}$$

$$= \text{N} \times \text{strain.}$$

$$\therefore \frac{2 \times 2{,}240}{33 \times 90{,}000} = \text{strain.}$$

$$\text{Strain} = \frac{\text{motion}}{\text{distance from fixed place}}.$$

$$\text{Strain} \times 12 = \text{motion in inches.}$$

$$\frac{2 \times 2{,}240 \times 12}{33 \times 90{,}000} = \text{motion} = \cdot 018 \text{ inch.—}\textit{Ans.}$$

7. Diameter, ·293 inch.

LESSON XVIII.

STRENGTH OF BOILERS AND PIPES.

Rule for strength of cylindric shell:—The pressure, per square inch, of the fluid inside × diameter of boiler or pipe in inches = 2 × thickness of metal × stress material will bear in lb. per square inch.

The general rule is:—"The area of cross-section of vessel, in plane of fracture, in square inches × pressure of fluid per square inch = area of metal laid bare by fracture × stress material will stand in lb. per square inch." To show that this is so, imagine the vessel shown in Fig. 47 to be subjected to internal fluid pressure. Let it be mounted on frictionless wheels; then, if we imagine a closely-fitting door inserted at B C, the vessel will not tend to move as a whole. Now let the fluid in the left-hand end escape, the vessel still remains at rest. But the total force to the left (at right angles to B C) is equal to the pressure of the fluid in the end B C D, per square inch × the area of the door in square inches, hence this is also the amount of the resultant pressure in the *opposite* direction, due to the

fluid acting on the curved end B C D. This is the force *causing* bursting at B C, the force *resisting* bursting is $a\ f$, where a is the area of metal which would be laid bare by fracture at B C, and f is the ultimate stress of the metal in lb. per square inch. Hence the rule. This rule is only applicable when the diameter of the vessel is large in comparison with the thickness of the metal. Applying this rule to the case of cylindric boilers, bursting at a longitudinal diametral plane, the change in strength due to the ends being neglected, the length and diameter being l and d respectively, the area of metal

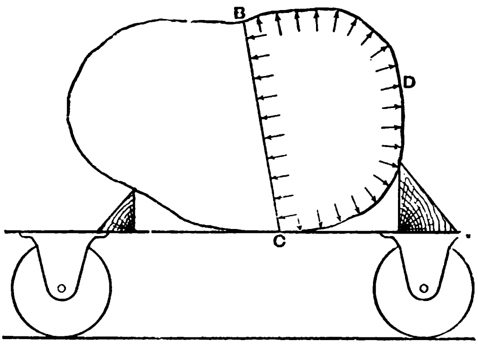


Fig. 47.

laid bare by fracture would be $2\ l\ t$, and the force causing bursting $p\ l\ d$, whence $p\ d = 2\ t\ f$, t being the thickness of metal and f the breaking tensile stress of the material. At a plane at right angles to the length of the boiler the quantities are $\pi\ d\ t\ f$, and $p\ \frac{\pi}{4}\ d^2$ respectively; in this case the rule would be $p\ d = 4\ t\ f$, hence the boiler is twice as likely to burst in the former way as the latter.

Numerical Examples.

1. A cylindric boiler is 6 feet in diameter, the thickness of metal being $\frac{5}{8}$ inch. Find the greatest safe pressure of the steam, neglecting weakness due to

riveted joints. Safe stress of the material in tension, 10,400 lb. per square inch.

2. If the greatest safe pressure in a boiler is to be 120 lb. per square inch, the diameter of the boiler 6 feet 6 inches, the strength of the riveted joints ·58 of that of the plates, the safe stress of the material as in the last Example, find the proper thickness of metal.
3. Find the bursting pressure of a spherical boiler 6 feet in diameter, the thickness of the metal being $\frac{1}{2}$ inch, and joints neglected. Breaking stress in tension, 50,700 lb. per square inch. If the riveted joints diminish the strength of the boiler 25 per cent., find the bursting pressure. Rule, $p\,d = 4\,t\,f$.
4. In hydraulic mains the inside diameter is 6 inches, working pressure of water 700 lb. per square inch, safe stress of the metal 3,000 lb. per square inch. Find the thickness of metal required.

RULE for *thick* cylinders: $p\,\frac{(D^2 + d^2)}{(D^2 - d^2)} = f$, where p is the pressure of the fluid inside, and f the stress the material will stand, both in lb. per square inch, D and d being the external and internal diameters respectively, measured in inches. The rule may be written $D = d\sqrt{\frac{f+p}{f-p}}$.

5. If the pressure is twice that given in the last Example, find the thickness of metal.
6. The pipes from an accumulator have to convey water at a pressure of 1,200 lb. per square inch. If the internal diameter of the pipes is 6 inches, find the least external diameter consistent with safety.
7. Using the rule for boilers, find the "head" of water which will be borne by water-mains 4 feet in diameter and 1 inch thick, the safe stress of the metal being 3,000 lb. per square inch, and $\frac{1}{8}$ inch being deducted for want of concentricity in casting.

RULE.—"Head" in feet $\div$ 2·3 = pressure of water in lb. per square inch. The term "head" means the vertical height of the surface of the water in the reservoir above the level of the pipe.

8. Taking the practical rule for water-mains $t = \frac{h \times d}{13{,}000} + \frac{1}{8}$, where t is the thickness in inches, h the head in feet, and d the diameter of the pipe in inches, find the head of water which will be borne by mains 4 feet in diameter and $\frac{1}{2}$ inch thick.
9. Find the thickness of pipes, 3 feet in diameter, head 250 feet.

ANSWERS.

1. Safe pressure, 180·5 lb. per square inch.
2. Thickness, ·77 inch.
3. Bursting pressure, 1,408·3 lb. per square inch. If joints considered, bursting pressure will be 1,056 lb. per square inch.
4. Thickness, 0·8 inch.
5. Thickness, 1·97 inch.
6. Outside diameter, 9·16 inches.
7. Head, 251·6 feet.
8. 101·56 feet.
9. Thickness, ·817 inch.

LESSON XIX.

STRENGTH AND STIFFNESS OF SHAFTS.

FIG. 48 shows a small portion—unit length—of a shaft subjected to torsion. One section may be supposed fixed and the other twisted round relatively to it, one end of a little column of the stuff moving in the arc of a circle about O as centre.

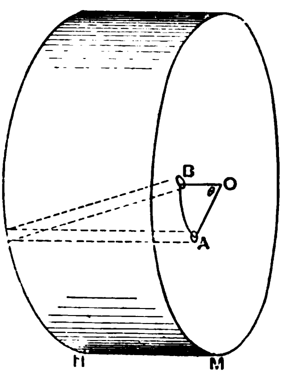


Fig. 48.

The angle moved through relatively to O is the angle A O B = θ radians, say. Let the end of the little column be a square inches in area (a being an exceedingly small fraction); then, if f is the

safe shear stress of the material, $f\,a$ is the greatest tangential force we may suppose permissible on the end of the little column. The moment of this force about the centre of the shaft is $f\,a \times r$, where r is the radius A O or O B of the path described by a; and the sum of all such products as this, for all the little columns into which we can conceive the shaft to be divided, is equal to the total torque applied to the shaft.

The applied twisting moment, or torque, we may denote by M_t. We have at once therefore the law—

$$(1)\ M_t = \Sigma f\,a\,r.$$

The distance moved by the end of the little column is A B $= r\,\theta$, and shear strain is—

$$\frac{\text{Distance moved}}{\text{Distance from fixed place}} = \frac{r\,\theta}{1}.$$

Hooke's law tells us that—

$$\text{Shear stress} = \text{N} \times \text{shear strain.}$$
$$\therefore \text{Shear stress} = \text{N}\,r\,\theta.$$

We have represented shear stress by f in equation (1) above, but we can now put for f its value N $r\,\theta$, which gives us—

$$\begin{aligned} M_t &= \Sigma\, \text{N}\, r\,\theta\, a \times r \\ &= \Sigma\, \text{N}\, a\,\theta\, r^2 = \text{N}\,\theta\,\Sigma\, a\, r^2. \end{aligned}$$

An expression similar to $\Sigma\, a\, r^2$ has been already explained. It is the moment of inertia of the section about the axis from which r is measured.* Hence—

$$(2)\ M_t = \text{N}\,\theta\,\text{I},$$

which is the law for the stiffness of a shaft. I for a circle (which is the shape of the section in this case, and in the case of most shafts) about a line through its centre perpendicular to its plane is $\frac{\pi\, d^4}{32}$, d being the diameter of the circle.

* The moment of inertia in all questions of torsion is about a line *at right angles* to the section, and is often called the "polar" moment of inertia. It is different from that used in questions on bending which is taken about a line *in the section*.

The law, therefore, becomes—

$$(2a)\ \theta = \frac{32\ M_t}{\pi\ N\ d^4}.$$

which will be best understood if read as follows:—*The angle of twist per unit length of a cylindric shaft of diameter d inches, when subjected to a twisting moment of* M_t *pound-inches, is equal to* 32 *times the twisting moment divided by the product of* π, *the fourth power of the diameter and the modulus of rigidity of the material.*

For hollow cylindric shafts $I = \frac{\pi}{32}(D^4 - d^4)$; hence, the rule becomes, for such a shaft—

$$\theta = \frac{32\ M_t}{\pi N (D^4 - d^4)}$$

D being the external and d the internal diameter.

The rule for the strength of a shaft is now easily obtained. We saw that the greatest shear stress on the little column is $N\ r\ \theta$, or $f_s = N\ r\ \theta$, where f_s is the proof or safe shear stress of the material, as required. This may be written $N \frac{d}{2} \theta = f_s$. Put into this for θ, its value as given in equation (2 a), and we have—

$$N \frac{d}{2} \times \frac{32\ M_t}{\pi\ N\ d^4} = f_s,$$

from which—

$$(3)\ M_t = \frac{\pi\ d^3}{16} f_s,$$

which is the rule for the *strength* of a solid cylindric shaft subjected to torsion only.

In hollow shafts it becomes—

$$M_t = \frac{\pi (D^4 - d^4)}{16\ D} f_s,$$

which means that a *solid cylindric shaft, d inches in diameter, will stand a torque of* M_t *pound-inches,* M_t *being of the numerical amount obtained by multiplying* π, *the third power of the diameter, and the proper shear stress of the material together, and dividing the product by* 16.

These rules hold only for stresses and strains below the elastic limit.

From this it is not difficult to deduce a practical rule for the diameter of a shaft which will safely transmit a given HP at a given speed. We have already seen (p. 64) that if a torque of T pound-feet act on a shaft revolving n times per minute, the HP transmitted is given by the rule :—

$$\mathrm{HP} = \frac{\mathrm{T} \times 2\,\pi\,n}{33{,}000}, \text{ whence } \mathrm{T} = \frac{33{,}000\ \mathrm{HP}}{2\,\pi\,n}.$$

The symbol M_t has been used to represent torque, or twisting moment, in *pound-inches;* hence,

$$\mathrm{T} = \frac{M_t}{12}, \text{ or } 12 \times \mathrm{T} = M_t.$$

Putting the value of M_t into rule (3), and taking 9,000 lb. per square inch as the safe shear stress of a wrought-iron shaft, we have—

$$\frac{12 \times 33{,}000 \times \mathrm{HP}}{2\,\pi\,n} = \frac{\pi\,d^3 \times 9{,}000}{16};$$

from which—

$$d^3 = \frac{12 \times 33{,}000 \times 16}{2 \times 3{\cdot}1416^2 \times 9{,}000} \times \frac{\mathrm{HP}}{n}, \text{ or } d = 3{\cdot}29\sqrt[3]{\frac{\mathrm{HP}}{n}},$$

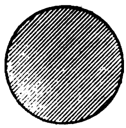
which may be taken as—

$$d = 3{\cdot}3\sqrt[3]{\frac{\mathrm{HP}}{n}}.$$

This is a very important practical rule. Taking the safe-working shear stresses of cast-iron and steel shafts to be respectively 4,500 and 13,500 lb. per square inch, we get the coefficients 4·1 and 2·88, instead of 3·3 for the rule just given, when that rule is applied to those materials. Remember these rules assume that the shaft is twisted *only*.

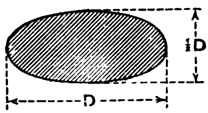
In the foregoing we have dealt only with cylindric shafts. If a right section of the shaft is *not* circular, the theory on which these calculations are based does not hold; in fact, a section which was plane before twisting occurred, no longer remains plane after strain, except in

the case of a circular section. For other shapes of section the theory is much too difficult for a work like this. The subject has been investigated by M. de St. Venant, and the results of his investigation are given in the article on Elasticity in the *Encyclopædia Britannica.* It is found * that sections have the relative values for resisting torsion shown by the numbers in Figs. 49, 50, 51, 52, 53, and 54, the sections being all of the same area, and the value of the circular section being taken as unity.



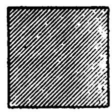
1·0

Fig. 49.



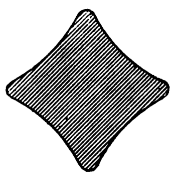
·80

Fig. 50.



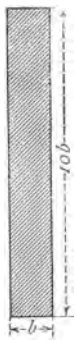
·88

Fig. 51.



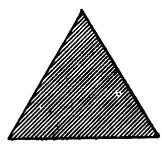
·87

Fig. 52.



·21

Fig. 53.



·73

Fig. 54.

Lord Kelvin has given a beautiful hydrodynamic illustration of the way in which the stress varies at various points on the boundary of a section. If we imagine a thin box made of the exact shape of the shaft, and filled with a frictionless fluid, then if the box be suddenly rotated about its axis, the *velocity of the fluid*

* The student should consult the article referred to.

relative to the box at any point represents the shear stress in a shaft of a similar shape when twisted.

In most non-circular sections it will be seen from this rule that the stress at any point on the surface of the shaft is greatest nearest the centre.

It should be borne in mind that shafts in practice are subjected to *bending* as well as twisting, owing to the loads due to the weights of pulleys and pulls of belts.

This matter is referred to more fully in Lesson XXIII., but the result is that the diameter of the shaft has to be *increased* by an amount depending on the amount of bending which the shaft has to withstand.

Thus, if the practical rule for the diameter of a wrought-iron shaft is $d = 3{\cdot}3\sqrt[3]{\frac{\text{HP}}{n}}$ when torsion only is considered; it will be $d = c \times 3{\cdot}3\sqrt[3]{\frac{\text{HP}}{n}}$ when bending is taken into account. Some values of c are given below:—

Kind of Shaft.	Value of c.
Propeller shafts of steamships, and shafts with similar load	1·13
Line-shafting in mills, etc.	1·3
Crank-shafts and shafting subjected to shocks, such as shafts in some machine-tools, etc.	1·42

Thus, if a shaft is required for a mill or factory, if not subjected to excessive shocks or more than the usual amount of bending, its diameter (if it is of wrought-iron) would be found from the rule—

$$d = 1{\cdot}3 \times 3{\cdot}3\sqrt[3]{\frac{\text{HP}}{n}}.$$

Numerical Examples.—Part I.

In the following Examples the shaft is supposed to be subjected only to torsion.

1. Find the safe diameter of a wrought-iron shaft to transmit 60 HP at 120 revolutions per minute.

$$\text{The rule is } d = 3{\cdot}3\sqrt[3]{\frac{\text{HP}}{n}} = 3{\cdot}3 \times \sqrt[3]{\frac{60}{120}} \text{ in this case,}$$

$$= 33 \times \frac{1}{\sqrt[3]{2}} = \frac{3{\cdot}3}{1{\cdot}26} = 2{\cdot}62 \text{ inches.}$$

2. A wrought iron shaft, 3 inches in diameter, rotates 150 times per minute; what HP will it transmit with safety?
3. If a shaft transmits safely 100 HP at the speed of 150 revolutions per minute, what power will it transmit with safety at 200 revolutions per minute?
4. Find the twisting moment necessary to produce in a wrought-iron shaft, $1\frac{1}{2}$ inch in diameter and 12 feet long, a twist of 12°. If this shaft revolves 150 times per minute, what HP will produce the same twist? N = 10,500,000.

   NOTE.—A twist of 12° in 12 feet is a twist of $\frac{1}{12}$° in 1 inch, or $\frac{1}{12} \times {\cdot}0175$* radian in 1 inch, $= \theta$ of rule 2a.
5. Will the twisting moment found in the last Example be too great for the shaft to transmit with safety, the safe shear stress being taken as 9,000 lb. per square inch?
6. A solid cylindric shaft is 5 inches in diameter; find the external diameter of a hollow shaft of the same material, the internal diameter of which is two-thirds of its external diameter, and which shall have (*a*) the same *strength*, (*b*) the same *stiffness*, as the solid shaft.

   NOTE.—By equating the expressions for the strengths of a hollow and a solid shaft we get the rule required for (*a*), the rule for (*b*) being obtained by equating the respective expressions for their stiffnesses.

Numerical Examples.—Part II.

In solving the following Examples the rules which allow for bending as well as twisting are to be taken.

* The number of degrees in any angle $\times \frac{\pi}{180}$ (which is approximately ·0175) gives the number of radians in the angle.

1. Find the diameter of a wrought-iron mill-shaft, to transmit 250 HP at 200 revolutions per minute.
2. A wrought-iron crank-shaft is 6 inches in diameter, and rotates 96 times per minute; what HP will it transmit safely?
3. Find the diameter of a solid steel propeller shaft to transmit 12,000 HP at 80 revolutions per minute.
4. If the shaft in the last Example is to be hollow, find its external diameter from *strength* considerations—(*a*) when its internal diameter is three-fourths of its external diameter, (*b*) when its internal diameter is two-thirds of its external diameter.

NOTE.—For hollow steel shafts subjected to torsion only, the strength rule simplifies to $\frac{\text{HP}}{n} = \cdot 042 \frac{D^4 - d^4}{D}$. For wrought-iron shafts use ·028 instead of ·042. If bending is taken into account, multiply the HP by c^3.

5. A steel shaft in a certain machine-tool has to transmit 10 HP at 100 revolutions per minute; find its proper diameter.

ANSWERS.

PART I.

2. 112·7 HP. 3. $133\frac{1}{3}$ HP.
4. 7,619 pound-inches. 18·14 HP. 5. Yes.
6. D = 5·38 inches. D = 5·28 inches.

PART II.

1. Diameter, 4·62 inches. 2. 201·5 HP.
3. Diameter, 17·29 inches.
4. (*a*) External diameter, 19·68 inches. (*b*) External diameter, 18·59 inches.
5. Diameter, 1·898 inch.

LESSON XX.

STRENGTH OF BEAMS.

THE laws of bending are based on the assumption that sections which were *plane sections remain plane* after the beam is bent. This gives us immediately the result that

in cross-sections the strain to which any little portion is subjected is proportional to its distance from the *neutral line*, that being the elevation of the surface in which the strain is zero.

Let the area of the little column considered be a square inches, a being an exceedingly small fraction. Then, if the stress at 1 inch from neutral line is p, and that at distance y inches from the neutral line is f, evidently $f = p\,y$. The *force*—compressive on concave and tensile on convex side of beam—on the little column there is $a\,p\,y$, and the first condition for the equilibrium of a number of forces *not* acting through one point (see Lesson I., page 6) gives us the result—

$$\Sigma\, a\,p\,y = o, \text{ or } p\,\Sigma\, a\,y = o,$$

hence the *neutral line must pass through the centre of area of the section.*

The strength rule for a beam is obtained by applying the third law of the equilibrium of forces, and equating the algebraic sum of the moments of the internal resisting forces to that of the external forces acting on the beam. Taking moments about the neutral line in a section—

$$\Sigma\, a\,p\,y \times y = \text{M},$$
$$\therefore p\,\Sigma\, a\,y^2 = \text{M},$$
$$\text{or } \frac{f}{y}\,\text{I} = \text{M},$$

where M is the "bending moment" applied to the beam, and I the moment of inertia of the section about the neutral line. The bending moment is obtained by taking the algebraic sum of the moments of all the external forces to *one* side of the section about the line referred to. I $\div$ y has been called the *strength modulus* of the section, and is generally denoted by the letter Z, the rule being that a beam will stand a bending moment at a given section whose amount is obtained by *multiplying the greatest stress the material will stand by the strength modulus of the section.* One inch being the unit of length, the bending moment will be expressed in pound-inches.

Values of the moments of inertia and strength moduli of some common sections are given in the following table:

TABLE I.

Moments of Inertia and Strength Moduli of Sections.

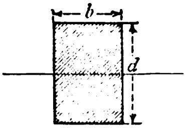
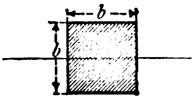
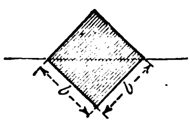
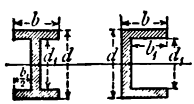
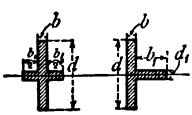
Section.	Moment of Inertia about Axis shown through centre of Area.	Strength Modulus of Section.
 Rectangle.	$\frac{bd^3}{12}$	$\frac{bd^2}{6}$
	$\frac{b^4}{12}$	$\frac{b^3}{6}$
 Square.	$\frac{b^4}{12}$	$0{\cdot}118b^3$
	$\frac{bd^3 - b_1d_1^3}{12}$	$\frac{bd^3 - b_1d_1^3}{6d}$
 Combinations of Rectangle.	$\frac{bd^3 + b_1d_1^3}{12}$	$\frac{bd^3 + b_1d_1^5}{6d}$

TABLE I. (*continued*).

Moments of Inertia and Strength Moduli of Sections.

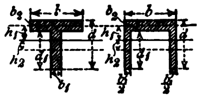
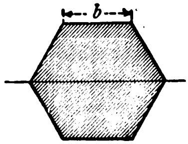
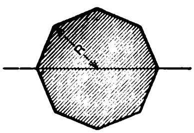
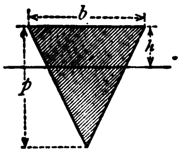
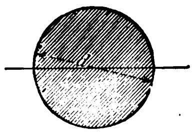
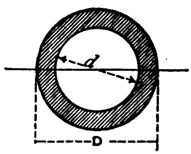
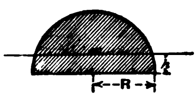
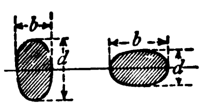
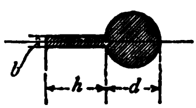
Section.	Moment of Inertia about Axis shown through centre of Area.	Strength Modulus of Section.
 Combinations of Rectangle.	$bb_2\left(\frac{b_2^2}{12}+h_1^2\right)$ $+ b_1d_1\left(\frac{d_1^2}{12}+h_2^2\right).$ $h_1 = \frac{b_1d_1(d_1+b_2)}{2(bb_2+b_1d_1)}.$ $h_1+h_2=\frac{d_1+b_2}{2}.$	
 Hexagon.	$0{\cdot}442b^4$	$0{\cdot}514b^3$
 Octagon.	$0{\cdot}638R^4$	$0{\cdot}69R^3$
 Triangle.	$\frac{bp^3}{36}.$ $\left(h=\frac{p}{3}\right).$	$\frac{bp^2}{24}$

TABLE I. (*continued*).

MOMENTS OF INERTIA AND STRENGTH MODULI OF SECTIONS.

Section.	Moment of Inertia about Axis shown through centre of Area.	Strength Modulus of Section.
	$\frac{\pi d^4}{64}$. $\frac{\pi}{64} = \cdot 0491.$	$\frac{\pi d^3}{32}$. $\left(\frac{\pi}{32} = \cdot 0982\right)$
 Circle.	$\frac{\pi}{64}(D^4 - d^4)$	$\frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right)$
 Semicircle.	$0 \cdot 11R^4$ $\left(h = \frac{4}{3}\ \frac{R}{\pi} = \cdot 424R\right)$	$0 \cdot 191R^3$
 Ellipse.	$\frac{\pi}{64}bd^3$	$\frac{\pi}{32}bd^2$
	$\frac{\pi d^4}{64} + \frac{hb^3}{12}$	

The strength rule for beams cannot be assumed for loads exceeding the elastic strength of the beam, but it has been found by experiment with beams of simple—say rectangular—sections that the load, W lb., required to break a beam l inches long, b inches broad, and d inches deep, is proportional to bd^2, and inversely proportional to l.

Hence, for such sections, and with materials such as timber, the simple strength rule—

$$W = c \times k \times \frac{bd^2}{l},$$

may be employed, c being a constant depending on the method of loading and supporting the beam, and k a constant depending on its material.

Values of c are given in Table II., and values of k in Table III.

TABLE II.

RELATIVE STRENGTHS OF BEAMS, SUPPORTED AND LOADED AS INDICATED.

Method of Loading and Supporting the Beam.	Greatest bending moment. Total load in each case = W, and length of beam = l.	Relative Strength. c.	Relative Deflection. D.
Fixed at one end and loaded at the other	Wl	·25	16
Fixed at one end and loaded uniformly all along its length	$\frac{Wl}{2}$	·5	6
Supported at both ends and loaded in the middle ...	$\frac{Wl}{4}$	1	1
Supported at both ends and loaded uniformly	$\frac{Wl}{8}$	2	·625
Fixed at both ends and loaded at the middle ...	$\frac{Wl}{8}$	2	·25
Fixed at both ends and loaded uniformly	$\frac{Wl}{12}$	3	·125

TABLE III.

Material.	Value of k.	Value of S (deflection)
English oak	6,680	$\frac{1}{7\cdot2\text{ million}}$
„ ash	8,350	$\frac{1}{6\text{ million}}$
Teak	9,180	$\frac{1}{9\text{ million}}$
Pitch pine	6,520	$\frac{1}{8\cdot4\text{ million}}$
Red pine	5,350	$\frac{1}{5\cdot84\text{ million}}$
Yellow pine	4,480	$\frac{1}{6\cdot4\text{ million}}$
Norway spruce	5,600	$\frac{1}{6\cdot4\text{ million}}$
Beech	6,220	$\frac{1}{5\cdot4\text{ million}}$
Elm	4,050	$\frac{1}{4\text{ million}}$
Mahogany	6,700	$\frac{1}{8\cdot5\text{ million}}$
Wrought iron	30,500	$\frac{1}{112\text{ million}}$
Cast iron	20,500	$\frac{1}{68\text{ million}}$
Cast brass	11,000	$\frac{1}{44\text{ million}}$

Numerical Examples.

1. A beam of English oak, 25 feet long, 10 inches broad, and 14 inches deep, is supported at the ends and loaded at the centre. Find its safe load, using 6 as a factor of safety.

The rule is—

$$W = c \times k \times \frac{bd^2}{l};$$

in this case—

$$W = \frac{1 \times 6{,}680 \times 10 \times 14^2}{25 \times 12}, \text{ or } 43{,}642 \text{ lb.}$$

Hence the safe load is—

$$\frac{43{,}642}{6} = 7{,}273 \cdot 6 \text{ lb.}$$

2. A pitch-pine beam, 30 feet long, 15 inches deep, and 12 inches broad, is fixed at the ends and loaded uniformly. Find the safe total load, using the same factor of safety as before.
3. A floor 24 feet square is supported by a red pine beam, which is fixed into the walls. Supposing the beam to support the whole weight, that the flooring, etc., weighs 20 lb. per square foot, and that the room is to accommodate 120 persons, weighing on an average 120 lb. each—find the proper section for the beam, its breadth being two-thirds of its depth. Factor of safety as before.
4. In the last Example, if the factor of safety for the dead load is 5, and that for the live load 10, find the proper section for the beam.
5. A timber beam is supported at points 12 feet apart, and loaded with weights of 10, 12, and 8 cwt. at points 2, 5, and 9 feet respectively from the left-hand support; find the bending-moment at a section midway between the supports. If this bending-moment were produced by a load at the centre of the beam, find the amount of that load and the proper size for the beam, it being of teak, and its breadth $\frac{5}{8}$ of its depth. Factor of safety, 6.
6. A wrought-iron beam is of the section shown in the fourth figure of Table I., the breadth b being 8 inches, depth d 12 inches, and thickness of metal everywhere 1 inch. If the beam is 25 feet long, and supported at the ends, find the greatest uniformly-distributed load it will bear with safety, safe f being 9,000.
7. An iron beam is of the shape of a hollow cylinder, the outside diameter being 10 inches, and thickness of metal $1\frac{1}{2}$ inch. If the beam is fixed firmly into two walls 30 feet apart, find the greatest uniform load it will bear with safety, safe f being 9,000.

8. A beam of the same material is semicircular in section, the circle being 8 inches in diameter. It is supported, flat side downwards, at points 12 feet apart, and loaded uniformly. Find the greatest safe load.
9. A beam of uniform section is a hollow cylinder, its external diameter being 1 foot, and thickness of metal 1 inch. Find the load at the centre which it will bear with safety, it being 25 feet long and *fixed* at the ends, safe f being taken at 8,000.
10. A continuous timber beam is supported by three supports 15 feet apart, and loaded with a uniform load of 5 cwt. per foot run. Find the section of the beam at a point 2 feet from the centre support, its breadth being $\frac{5}{8}$ of its depth. Safe $f = 1{,}500$.
11. A similar and similarly-loaded beam is supported by four equidistant supports 15 feet apart. Find the section of the beam at the centre of the middle span.

NOTE.—For a beam continuous over three equidistant supports, the reactions at the supports are, in lb.—

$$\tfrac{3}{8}\,wl,\ \tfrac{5}{4}\,wl,\ \text{and}\ \tfrac{3}{8}\,wl;$$

in a beam continuous over four equidistant supports, the reactions are—

$$\tfrac{4}{10}\,wl,\ \tfrac{11}{10}\,wl,\ \tfrac{11}{10}\,wl,\ \text{and}\ \tfrac{4}{10}\,wl,$$

where w is the load in lb. per inch run, l the length of one span in inches.

12. A rolled iron beam of double T-section is 25 feet long, supported at the ends, and has to bear a load of $\frac{1}{2}$ ton per foot run; if the depth is to be $\frac{1}{20}$ of the span, find the area of the top flange of the beam approximately, the web being neglected. Safe compressive stress, 4 tons per square inch.

ANSWERS.

2. 24,450 lb. 3. $d = 16{\cdot}12$ inches. $b = 10{\cdot}74$ inches.
4. Depth, 17·57 inches. Breadth, 11·72 inches.
5. Bending moment at centre, 69,888 lb.-inches. Equivalent load at centre, 1,941·3 lb. Safe section is of depth, 6·63 inches, breadth, 4·14 inches.

6. 22,746 lb. ($I = 568\cdot6$) ($Z = 94\cdot76$). 7. 22,381 lb.
8. 6,112 lb. 9. Safe load, 18,737 lb.
10. Depth, 7·9 inches. Breadth, 4·94 inches.
11. Depth, 6·23 inches. Breadth, 3·89 inches.
12. Area = 7·81 square inches. In this case do not find the moment of inertia of the section. Take moments of all external forces to left (say) of section, about a point in lower flange. Equate the algebraic sum of these moments—*i.e.*, the "bending moment"—to the moment of the safe compressive force in the top flange about the same point. This gives the safe compressive force in the top flange = AP lb. say, where P is the given safe compressive stress, and A the area of flange in square inches. Hence A is found. This method is usually adopted in practice in designing rolled beams; though not so accurate as the rule given on page 102, it is much easier. The rule really amounts to this:—"Total force on one flange $= \frac{M}{d}$," M being the bending moment, and d the depth of the beam.

LESSON XXI.

STIFFNESS OF BEAMS. CHANGE OF CURVATURE AND DEFLECTION.

THE strain in the case of a loaded beam is such that the beam if originally straight becomes curved, and if originally curved has its curvature increased or diminished, according as the *added* bending moment acts with or against that already present. "Curvature" is defined as the *reciprocal of the radius of curvature*, this radius being the radius of the circle agreeing most nearly with the curve at the point indicated.

In Fig. 55 a small portion of a bent or curved beam is shown, the curvature being greatly exaggerated. Take O O′ (measured along the neutral line) = 1 inch, and O A = 1 inch; this will simplify our expressions. From the similarity of the sectors O O′ C and A B C it is evident that $\frac{O\,O'}{O\,C} = \frac{A\,B}{A\,C}$. Let O C be called r (the radius of

curvature), then $A\,C = r + 1$. Also, $A\,B = A\,N + N\,B = 1 +$ the strain at 1 inch from the neutral line.

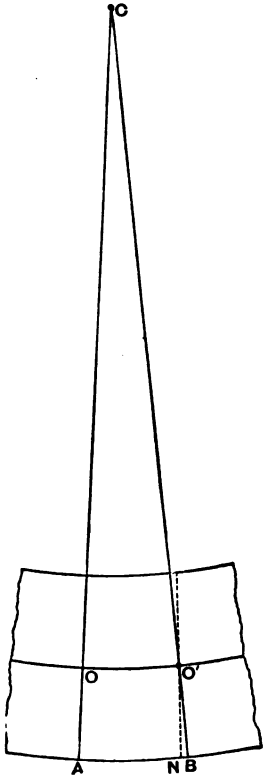


Fig 55.

It has already been shown that the *stress* at 1 inch from the neutral line is $\frac{M}{I}$; and since $\frac{\text{stress}}{E} =$ strain, the strain at A B must be $\frac{M}{E\,I}$, and $A\,B = 1 + \frac{M}{E\,I}$.

Hence, $\frac{O\,O'}{O\,C} = \frac{A\,B}{A\,C}$ may be written—

$$\frac{1}{r} = \frac{1 + \frac{M}{E\,I}}{r + 1};\ i.e.,\ \frac{r+1}{r} = 1 + \frac{M}{E\,I} \text{ or } \frac{1}{r} = \frac{M}{E\,I}$$

If the beam was originally curved to a radius r_0, the addition of the bending moment M produces a change of curvature.

$$\frac{1}{r} - \frac{1}{r_0} = \frac{M}{E\,I}.$$

In regard to the actual deflection of any point in a horizontal beam, it is only necessary here to refer to one case, viz., that of a beam supported at both ends and loaded uniformly. Let deflection be represented by y, and distance from one support by x. Then, in works on the Differential Calculus, it is proved that—

$$\frac{1}{r} = \frac{\frac{d^2 y}{d x^2}}{\left\{1 + \left(\frac{d y}{d x}\right)^2\right\}^{\frac{3}{2}}}.$$

Neglecting the very small quantity $\left(\frac{d y}{d x}\right)^2$ we have (since $\frac{1}{r} = \frac{M}{E\,I}$)—

$$\frac{d^2 y}{d x^2} = \frac{M}{E\,I}.$$

If w lb. per unit of length is the intensity of load on the beam whose length is l, then $M = \frac{w\,l\,x}{2} - \frac{w x^2}{2}$, and if $x = \frac{l}{2}$, the bending moment and deflection are both a maximum; hence the deflection at the centre is obtained by integrating twice the expression—

$$\frac{d^2 y}{d x^2} = \frac{w\,l^2}{8\,E\,I}.$$

The result is that the deflection $y = \frac{5}{8} \times \frac{W\,l^3}{48\,E\,I}$, W being the total load on the beam or $w\,l$. If the beam bore the

same total load, but concentrated at its middle point, the deflection would be that given in the expression above, but without the factor $\frac{5}{8}$. The relative deflections of beams loaded and supported in different ways are given in the fourth column of Table II., page 106. Hence, for any of the beams referred to,—

$$y = D \times \frac{W\, l^3}{48\, E\, I}.$$

For beams of rectangular section the rule may be written simply thus—

$$y = D \times S \times \frac{W l^3}{b\, d^3};$$

values of S being given in Table III. In all this one inch is the unit of length.

Numerical Examples.

1. Find the deflection at the centre of a beam of English oak, 30 feet long, 15 inches deep, and 10 inches broad, supported at the ends, and loaded at the centre with a load of 5,000 lb.
2. Find the greatest deflection of a pitch-pine beam, 25 feet long, 14 inches deep, and 9 inches broad, fixed at the ends, and loaded at the centre with $\frac{1}{5}$ of its breaking load.
3. A solid cylindric wrought-iron shaft, 3 inches in diameter, is supported at points 16 feet apart, and loaded at the centre with a load of 400 lb. Find the deflection of the shaft due to this load and to its own weight. A cubic inch of wrought-iron weighs ·28 lb., and E may be taken as = 28,000,000.

   NOTE.—Find the deflection due to each load separately, and add the results to get the total deflection.
4. A teak-beam, 20 inches square, is fixed firmly into walls 30 feet apart, and loaded uniformly. Find its greatest safe load, and the deflection under this load. Factor of safety, 5.
5. A wrought-iron beam of T-shaped section has the following dimensions:—breadth of top flange, 5

inches; depth of web, 6 inches; thickness of metal everywhere, $\frac{3}{4}$ inch. The beam is 16 feet long, and fixed at the ends. Find the deflection under its greatest safe, uniformly-distributed load, and the amount of that load. Safe f, 9,000; E, as in Example 3.

6. A beam of English oak, the section of which is 12·37 inches deep and 7·73 inches broad, is supported at

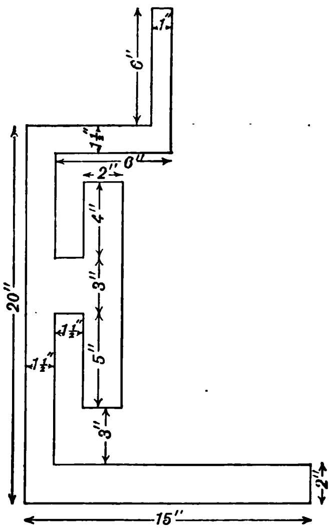


Fig. 56.

points 16 feet apart, and bears a uniform load amounting to $5\frac{1}{4}$ tons. Find the deflection of the beam at the middle point of its length.

7. A rolled iron beam, the moment of inertia of the section of which is 568·6, is supported at points 25 feet apart, and loaded with a uniformly-distributed load of 22,744 lb. Find its greatest deflection, E being 29,000,000 lb. per square inch.

8. If l be the length of a round wrought-iron rod in feet,

and d its diameter in inches; if it is supported at the ends and loaded by its own weight merely, with a deflection of 1 inch for 100 feet of span; find the relation of l to d.

9. Find the moment of inertia and strength modulus of the accompanying section (Fig. 56) about a line through its centre of area parallel to the 15-inch side. Prove, if you can, the rule employed.

   RULE.—The moment of inertia of an area about a line *not through its centre* is equal to the moment of inertia *about a parallel line* through its centre *plus* the area multiplied by the square of the distance between the two lines.

ANSWERS.

1. ·96 inch. 2. Deflection, ·499 inch. Load, 15,335 lb.
3. ·844 inch. 4. Safe load, 122,400 lb. Deflection, ·496 inch.
5. Deflection, ·0816 inch. Safe load, 4,586·6 lb. ($I = 36{\cdot}97$.)
6. Deflection, ·4941 inch. 7. Greatest deflection, ·485 inch.
8. $l = 6{\cdot}21 \sqrt[3]{d^2}$.
9. Moment of inertia, 5,295·23. Strength modulus, 320·73; one inch being taken as the unit of length. (Distance of centre of area from lower edge, 9·49 inches.)

LESSON XXII.

STRENGTH OF THE TEETH OF WHEELS, ETC.

$N\ T = n\ t$ is the rule for velocities, where T is the number of teeth, and N the number of revolutions of one wheel per minute, n and t similar quantities for the other wheel gearing with it.

(1) $p = K \sqrt{P}$ is the rule for strength, where p is the pitch in inches, P the pressure between two wheels in lb., and K has the value ·05 for mill machinery, ·06 for machine tools, etc. (See Unwin's "Machine Design.") Also,

(2) $\dfrac{P \times v}{33{,}000} =$ HP transmitted, v being the velocity of the pitch circle of the wheel in feet per minute, evidently $= \dfrac{N\,T\,p}{12}$.

The whole pressure between the wheels does not, however, come on *one* pair of teeth; in fact the pressure on any one tooth is n P, where n is a fraction lying between $\frac{1}{2}$ and 1, and often taken $= \frac{2}{3}$. If, therefore, we take $\frac{2}{3}$ P as the pressure on one tooth, and suppose this pressure concentrated at one corner of the tooth—which may be considered as a beam fixed at one end and loaded at the other—we get the ordinary rules for the strength of wheel-teeth when the breadth of the tooth is not taken into account. The thickness of the tooth when new is ·48 p, but the tooth should be strong enough when worn, when its thickness may be only ·36 p.

Taking these things into consideration and combining (1) and (2) we get the rule (3) $T = c\,\dfrac{HP}{p^3\,N}$, where c has the values 990 for mill machinery, 1,425 for machine tools, etc.

In well-made wheels, such as are usually employed nowadays, the tooth may be supposed to touch fairly uniformly all along its face, and hence the breadth of the tooth must be taken into account. If the tooth is of breadth $= 2\frac{1}{2} \times$ pitch, the rule (3) becomes $T = 791\,\dfrac{HP}{p^3\,N}$ for mill machinery, the multiplier being about 1,140 for machine tools.

Numerical Examples.

1. A spur-wheel with 38 teeth, going at 140 revolutions per minute, gears with another which has 83 teeth. Find the speed of the second wheel.
2. In mill machinery a wheel with 48 teeth, pitch $1\frac{1}{4}$ inch, revolves 30 times per minute. What HP will it transmit safely? Teeth not supposed very well made.
3. A spur-wheel 18 inches in diameter, speed 40 revolu-

tions per minute, has to transmit 6 HP. Find the pitch. c as in last Example.

4. Find the lowest speed at which a spur-wheel of 16 inches diameter, $\frac{3}{4}$ inch pitch, in a machine tool, will transmit with safety 3 HP.

In the following Examples the wheels are supposed to be carefully made, so that a tooth may be assumed always to touch all across its face.

5. A carefully-made wheel in the machinery of a mill has 62 teeth of 1 inch pitch. Find the limiting speed at which it will transmit 5 HP.
6. A wheel, in a machine tool, with the same number of teeth has to transmit 6 HP at 40 revolutions per minute. Find the pitch and diameter.

ANSWERS.

1. 64·1 revolutions per minute. 2. 2·84 HP.
3. Pitch = 1·62 inch. 4. 151·2 revolutions per minute.
5. 63·79 revolutions per minute.
6. Pitch, 1·4 inch. Diameter, 27·68 inches.

LESSON XXIII.

COMBINATION OF TWISTING AND BENDING. CRANK-SHAFTS, ETC.

To find the resultant stress in a piece of material subjected at the same time both to twisting and bending, is one of the most important questions which can come before the engineer. A short investigation of this interesting problem will now be given. Consider a small block of material 1 inch at right angles to the paper (Fig. 57), the face A C being acted on by a shear stress of q lb. per square inch, and a tensile stress of p lb. per

square inch ; then, since shear stress in one set of planes is always accompanied by shear stress of equal intensity on planes at right angles,* the face B C will be acted on by the shear stress q lb. per sq. inch. The problem is to find what the angle θ is, so that the face A B may be acted on by a normal stress p_1 *only*, also what will be

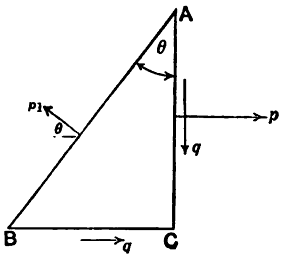


Fig. 57.

the amount of p_1. Let A C = 1 inch (the block should really be *very* small; imagine it to be one-millionth of an inch, and let everything be magnified one million times).

Equating horizontal components, $p + q$ B C $= p_1$ A B $\cos \theta$ (α).
Equating vertical „ p_1 A B $\sin \theta = q$ (β).
But A B $= \sec \theta$, B C $= \tan \theta$ $\therefore$ (α) and (β) become $p + q \tan \theta = p_1 \sec \theta \cos \theta = p_1$, and $p_1 \sec \theta \sin \theta = p_1 \tan \theta = q$.

$$\text{Hence } p_1 \tan \theta = q, \text{ or } \tan \theta = \frac{q}{p_1} \quad \ldots . (1).$$

$$\text{Also } p + q \tan \theta = p_1, \text{ or } p + q \frac{q}{p_1} = p_1.$$

$$\therefore p_1 p + q^2 = p_1^2.$$
$$p_1^2 - p_1 p = q^2.$$

And solving this quadratic, we have—

$$p_1 = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q^2} \quad \ldots . (2).$$

* This well-known law is proved in Thomson and Tait's "Natural Philosophy," Perry's "Mechanics," and other text-books.

Whence, combining with (1)—

$$\tan\theta = \frac{q}{\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q^2}} \quad \ldots \text{ (3).}$$

Hence, if a piece of material is acted on by a shearing stress of q lb. per square inch, and a tensile stress of p lb. per square inch at right angles to those interfaces subjected to the shear, the greatest stress to which the material is subjected is a tensile stress whose amount is $p_1 = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}$. Similarly, if we combine a compressive stress with a shear, we get the greatest resultant stress a compressive stress—$p_1 = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q^2}$. To apply this to the case of shafts, which are in all cases subjected to bending (*i.e.*, tension and compression), as well as torsion or shear, representing the twisting moment acting on a shaft by T, and the bending moment by M, we have the rules—

$$q = \frac{16\,T}{\pi\,d^3} \text{ and } p = \frac{32\,M}{\pi\,d^3}. \quad \text{(*See* pp. 96 and 102.)}$$

Putting these values into equation (2), we get—

$$p_1 = \frac{16\,M}{\pi\,d^3} + \sqrt{\left(\frac{16\,M}{\pi\,d^3}\right)^2 + \left(\frac{16\,T}{\pi\,d^3}\right)^2}$$

$$\text{(4)} \qquad \text{Whence } p_1 = \frac{16}{\pi\,d^3}\left\{M + \sqrt{M^2 + T^2}\right\}.$$

and since $\frac{p_1\,\pi\,d^3}{16}$ is of the form of a twisting moment, then for any material in which the greatest safe tensile and shear stresses are nearly alike, we have—

$$T\,e = M + \sqrt{M^2 + T^2} \quad \ldots \text{ (5),}$$

where T e is that twisting moment which is equivalent to the other two moments *combined.*

If we divide equation (4) by 32 instead of by 16, and remember that $\frac{p_1 \pi d^3}{32}$ is a bending moment, we have—
$M e = \frac{1}{2} \{M + \sqrt{M^2 + T^2}\}$, $M e$ being that bending moment which is equivalent to the moments M and T combined.

Numerical Examples.

1. A piece of wrought-iron is subjected to a tensile stress of 4 tons per square inch, and a shear stress of 3 tons per square inch, will the material be strained beyond the safe limit? Safe tensile stress, 10,400 lb. per square inch.
2. A piece of wrought-iron is subjected to a shear stress of 3 tons per square inch. Find the tensile stress which may be combined with it in order that the material may just be strained to the limit of safety.
3. A piece of wrought-iron is subjected to compressive and shearing stresses of equal intensity. Find the amount of each so that the material may just be subjected to its greatest safe stress. Safe stress as before.
4. A wrought-iron shaft is subjected to a bending moment of 10,000 lb.-inches, and a twisting moment of 20,000 lb.-inches. Find the twisting moment which will represent the combination of the two. Find also the safe diameter of this shaft. Safe stress in shear = 7,800 lb. per square inch.
5. A 3-inch wrought-iron shaft, making 150 revolutions per minute, is supported at wall-brackets 18 feet apart. Midway between these bearings a belt is driven from a pulley on the shaft, the resultant side-pull due to the weight of the pulley and pull of the belt being 200 lb. What is the greatest HP the shaft will transmit with safety, the weight of the shaft being neglected?
6. A 3-inch shaft transmits 68 HP at 150 revolutions per minute; the bearings of the shaft are 16 feet apart, and 3 pulleys are keyed on the shaft—one at

the centre, and the other two 5 and 6 feet respectively from the bearings. If we know that the resultant side-pull is the same for each pulley, find its amount so that the shaft is just strained to the safe limit.

7. The spindle of an electro-motor runs at 1,000 revolutions per minute, and the motor receives a current of 40 amperes, at an E.M.F. of 100 volts. Find the diameter of the wrought-iron spindle, if the bending moment produced by the pull of the belt on the overhung pulley is equal to the twisting moment, and the efficiency of the motor be taken at 70 per cent.

8. In the case of certain kinds of shafting it is possible to express M as a multiple of T. Thus $M = k\,T$.

   Rankine has found the following values of k:—

For such cases as propeller-shafts... ...	k varies from	·25 to	·5.
For ordinary light shafting in mills ...	k „	·75 to	·1.
And for crank-shafts and other heavy shafting	k „	1·0	to 1·5.

   For each of the cases above mentioned deduce the constant a in the simple formula for the diameter of a shaft, $d = a\sqrt[3]{\dfrac{HP}{n}}$.

   NOTE.—The values here found agree with values of the multiplier c given at page 99.

9. Find the two equal stresses (tensile and shear) which *combined* will produce a resultant tensile stress of 66,000 lb. per sq. inch.

10. A cylindric shaft transmits 30 HP at 150 revolutions per minute, and is subjected to a bending moment equal to the twisting moment; find the diameter of the shaft, if the safe shear stress of the material is 9,000 lb. per sq. inch.

ANSWERS.

1. Yes. The resultant stress is 5·6 tons per square inch.
2. 6,057·8 lb. per square inch.
3. 6,428 lb. per square inch

4. $Te = 32{,}360$ lb.-inches. $d = 2{\cdot}76$ inches.
5. 68 HP. Substitute for T its value in terms of HP, thus

$$\frac{\frac{T}{12} \times 2\pi n}{33{,}000} = \text{HP, find } T_e \text{ from rule (5), p. 119, then}$$

$$Te = \frac{\pi d^3 f}{16}, \text{ from which } d \text{ is found.}$$

6. 94 lb. each. 7. Diameter of spindle, ·708 inch.
8. 1st case, $d = 3{\cdot}7$ to $3{\cdot}98\sqrt[3]{\frac{HP}{n}}$.

2nd case, $d = 4{\cdot}28$ to $4{\cdot}5\sqrt[3]{\frac{HP}{n}}$.

3rd case, $d = 4{\cdot}5$ to $5\sqrt[3]{\frac{HP}{n}}$.

9. 40,791·1 lb. per square inch.
10. Diameter, 2·58 inches.

LESSON XXIV.

OVERHUNG CRANK. CRANK-PINS. LENGTHS OF BEARINGS, ETC.

A VERY good illustration of combined twisting and bending actions is the case of an overhung crank (Fig. 58). Let P be the total push, or pull, of the connecting-rod at right angles to the direction of the crank.

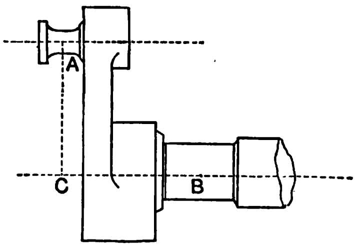


Fig. 58.

Then $T = P \times AC$, and $M = P \times BC$, the resultants of all the forces acting on the journal and crank-pin being assumed to pass through their centres respectively.

Then using our rule for combining twisting and bending moments, we have—

$$T\,e = P \times B\,C + \sqrt{(P \times B\,C)^2 + (P \times A\,C)^2}$$
$$= P\,\{B\,C + \sqrt{B\,C^2 + A\,C^2}\}$$
$$\therefore T\,e = P\,(B\,C + A\,B).$$

Hence we calculate the twisting moment for an overhung crank as if the crank were of a length equal to the horizontal distance B C, plus the sloping distance A B.

Numerical Examples.

1. The greatest pressure of steam in the cylinder of a non-condensing steam-engine, when the connecting-rod is at right angles to the crank, is 80 lb. per square inch, the diameter of the piston 12 inches, the length of the crank 12 inches, and the distance from a vertical line through the centre of the crank-pin to the centre of the crank-shaft journal 11 inches. Find the diameter of that journal, (*a*) taking the safe shear stress as 7,800, and (*b*) as 9,000 lb. per square inch.
2. The crank of a steam-engine is 15 inches long, the horizontal distance from the centre of the crank-pin to the centre of the nearest crank-shaft bearing is 12 inches. If the greatest push or pull of the connecting-rod is 15,000 lb., find the diameter of the crank-shaft, taking the safe shear stress of wrought-iron to be 9,000 lb. per square inch.

CRANK-PINS.

3. Rule for dimensions of crank-pin: $d = \cdot025\sqrt{\dfrac{P\,l}{d}}\cdot$
   The length of a crank-pin is $1\frac{1}{2}$ times its diameter, the diameter of the piston of the engine is 12 inches, maximum steam pressure 100 lb. per square inch. Find the dimensions of the crank-pin.

LENGTHS OF BEARINGS.

Many rules are given, none of which are applicable *generally*. The rule (1) . . . $\dfrac{l}{d} = 1 + \cdot004\ N$ is about one

of the best for ordinary shafting. N is the number of revolutions of the shaft per minute, l and d the length and diameter of the bearing in inches. This is independent of the pressure on the bearing. Another rule is (2) . . . $d\,l = \frac{1}{a}\{P + n\sqrt{P}\}$, which is applicable, for instance, to locomotive axles, P being the pressure on the bearing in lb., a in this case being 350.

NOTE.—Professor Perry has deduced the following rules for dynamo bearings:—

(1) $\frac{l}{d} = n\sqrt{L} \div 1{,}000$ for discoidal armatures; and

(2) $\frac{l}{d} = (n\sqrt{L} \div 3{,}500) + 2$ for drum and elongated ring armatures. In these rules L is the length of the shaft between the middle points of the bearings, and n the speed in revolutions per minute.

4. Find the length of the bearing for an ordinary shaft, 2 inches in diameter, going at 200 revolutions per minute.
5. Find the length of the bearing for a dynamo machine, the diameter of the journal being $2\frac{1}{2}$ inches, the pressure on each bearing 150 lb., and speed 800 revolutions per minute. First by rule (2), also by rule for dynamos, if armature is of drum-type. $L = 49$ inches.
6. In an Elwell-Parker dynamo (drum) the length between the centres of the bearings is 30 inches, and the speed 1,300 revolutions. Find the ratio of length of bearing to diameter.
7. In a "Victoria" dynamo (discoidal) the speed is 2,000, $L = 10$ inches. Find $\frac{l}{d}$.
8. In a Kapp dynamo (drum) $L = 46$ inches, speed 780, journal $2\frac{1}{2}$ inches in diameter. Find length of journal.
9. Ferranti alternator (discoidal), speed 1,700, $L = 28$ inches. Find $\frac{l}{d}$.

ANSWERS.

1. Diameter, 5·4 and 5·2 inches respectively.
2. 6·4 inches. 3. Length, 4·87 inches. Diameter, 3·25 inches.
4. $l = 3·6$ inches.
5. The first rule gives $l = 11·37$ inches. The dynamo rule gives $l = 8·6$ inches. In dynamos the bearings are usually of *unequal* length, that at the driving end being longer than the other.
6. $\frac{l}{d} = 4·03$. 7. 6·32.
8. Length, 8·77 inches. 9. 8·99.

LESSON XXV.

STRENGTH OF RIVETED JOINTS.

Consider the strength of a strip of plate held by one rivet, the joint being, say, a lap-joint. Fracture may occur in any one of the four ways shown in Fig. 59—

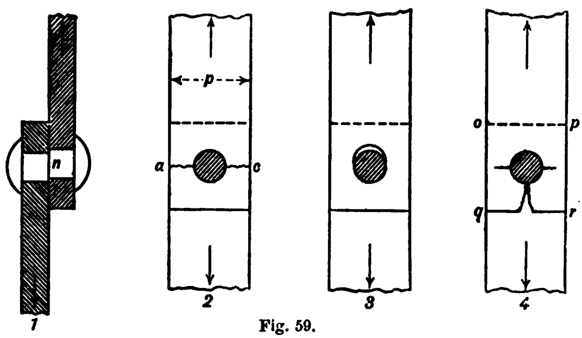


Fig. 59.

viz., (1) by the shearing of the rivet at n, (2) by the breaking of the plate by tension at a c, (3) by the crushing of one side of the hole, or (4) by the breaking of the plate as a beam fixed at the ends o p q r, and loaded at the middle.

Let d be the diameter of the rivet, p the pitch, t the thickness of the plates, f_t the ultimate tensile, f_c the ultimate compressive, and f_s the ultimate shear stress of the material in lb. per square inch, all dimensions being in inches. A rule might be obtained by equating the strengths of the joint under the four conditions shown, but it is found in practice that the joint is much more likely to give way either as shown in (1) or (2). In order that the strip may be equally ready to give way in each of these modes, we have the condition $\frac{\pi}{4} d^2 f_s = (p - d)\, t\, f_t$. (It is evident that the strip is p inches wide.)

But if we take $f_t = f_s$, which is nearly true for wrought iron and mild steel, and if we use the practical rule for diameter of rivet in terms of thickness of plate—

$$d = 1{\cdot}2 \sqrt{t},$$

then we have—

$$\frac{\pi}{4} d^2 = (p - d) \frac{d^2}{144},$$

$$\therefore p = \frac{1{\cdot}44}{4} + d = \text{A} + d.$$

It has been found practically that the distance from the side of the hole to the edge of the plate should not be less than d, else the joint may give way as in (4).

BUTT JOINT.

Consider one case of the butt joint with, say, two covering plates, as shown in Fig. 60. We will assume

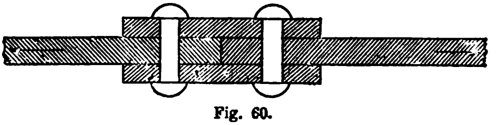


Fig. 60.

that each of the cover plates is just half the thickness of the main plate, but in reality it is usually a little more. The thickness of the main plates being as before t, then

equating the shearing resistance of the rivet—now in double shear—to the tensile resistance of the two cover plates, we have—

$$(p - d)\, t f_t = 2 \times \frac{\pi}{4} d^2 f_s,$$

and assuming

$$f_t = f_s \text{ as before, also } d = 1\cdot 2 \sqrt{t},$$
$$p = 2\cdot 406 + d = \mathrm{A} + d.$$

It will be seen from the foregoing how the values of A, given in the table below, have been found. It is not necessary here to enter into a discussion of the strength of other joints.

The strength of one strip of plate without hole $= p\,t\,f$, and the strength of one strip of plate with hole $=$ $(p - d)\ t\ f$, hence the strength of the boiler shell is diminished in the proportion $\frac{p-d}{p}$. Therefore, in boilers, if the strength of the joints is considered, the stress employed should be—

$$f = f_t \frac{p-d}{p} = f_t \frac{\mathrm{A}}{\mathrm{A} + d}.$$

For double and treble riveted joints, $f = k \frac{\mathrm{A}}{\mathrm{A} + d} f_t$, where k is nearly unity. Values of k are given in the table.

TABLE OF CONSTANTS—RECAPITULATION OF RULES, ETC.

(Compiled by Prof. Perry.)

For cylindric shell of boiler: (1) . . . $\mathrm{P} = \frac{f\,t}{r}$, where f is given in equation (3), t thickness, r radius in inches.

Same rule for flue, t being average thickness, taking strengthening rings into account; but in this case f is the compressive stress of the material.

(2) . . . $d = 1\cdot 2 \sqrt{t}$, d being diameter of rivet-hole, t thickness of plate.

Distance from edge of plate to side of hole, not less than d.

In any riveted joint, pitch $p = A + d$, where A is given in the following table :—

		Iron Plates and Iron Rivets.		Steel Plates and Steel Rivets.	
		Drilled Holes.	Punched Holes.	Drilled Holes.	Punched Holes.
Lap or butt joint, with one covering plate.	Single-riveted .	1·203	1·47	0·89	1·077
	Double- „ .	2·217	2·66	1·69	1·93
	Treble- „ .	3·23	—	2·49	—
Butt joint, with two covering plates.	Single-riveted .	2·406	2·94	1·78	2·154
	Double- „ .	4·434	5·32	3·38	3·86
	Treble- „ .	6·46	—	4·98	—

The actual value of f for tension to be employed is—

$$(3) \ldots f = k \frac{A}{A + d} f_t,$$

where f_t is tensile strength of untouched plate. A is given above, and k has the following values :—

Steel Plates.	Iron Plates.	
1·0	·88	Single-riveted, drilled.
0·9	·77	„ „ punched.
1·06	·95	Double- „ drilled.
1·0	·85	„ „ punched.
1·08	—	Treble- „ drilled.

In all this it is to be remembered that a punched hole is to be called a drilled hole if it has been rymered out or annealed after punching.

This gives rise to the following rule :—

$$d = \frac{a}{2} + \sqrt{\frac{a^2}{4} + a\,A} \ldots (4),$$

and where $a = \dfrac{1·44\, r\, P}{k\, A\, f_t}$.

d being diameter of rivets.
P, bursting pressure of boiler.
r, radius of boiler in inches.
f_t, ultimate tensile stress of unhurt plate.

Having found d, find t the thickness of plate by $t = d^2 \div 1{\cdot}44$.

Example.—A boiler is 5 feet 6 inches in diameter, the plates being of iron, $\frac{1}{2}$ inch thick, double-riveted lap-joints, holes punched, iron rivets. Calculate the diameter and pitch of the rivets.

If the ultimate stress of the plates is 46,000 lb. per square inch, find the greatest safe pressure of steam the boiler will bear, using 5 as a factor of safety.

$$p = A + d. \qquad d = 1{\cdot}2\sqrt{t}.$$

$$= 2{\cdot}66 + {\cdot}848. \qquad = 1{\cdot}2 \times \sqrt{\tfrac{1}{2}} = \frac{1{\cdot}2}{1{\cdot}414} = {\cdot}848 \text{ inch.}$$

$$\therefore\ p = 3{\cdot}508.$$

To find the safe pressure :—

First find the bursting pressure $P = \dfrac{f\,t}{r}$,

$$\text{where } f = k\,\frac{A}{A + d}\,f_t$$

$$= \frac{{\cdot}85 \times 2{\cdot}66}{3{\cdot}508} \times 46{,}000$$

$$f = 29{,}649 \text{ lb. per square inch.}$$

$$\therefore\ P = \frac{29{,}649}{33} \times \tfrac{1}{2} = 449{\cdot}22 \text{ lb. per square inch.}$$

Hence the greatest safe pressure is 89·84 lb. per square inch.

Numerical Examples.

1. Taking the ultimate stress of steel plates to be 62,000 lb. per square inch, calculate the diameter of rivets and pitch, for a double-riveted lap-joint, holes drilled in $\frac{1}{2}$-inch steel plates with steel rivets. If this is a boiler 6 feet in diameter, find the greatest safe pressure of steam inside it, using 4 as a factor of safety.
2. Find the same things for a single-riveted lap-joint, $\frac{1}{2}$-inch iron plates with iron rivets, holes drilled. Ultimate stress, 46,000 lb. per square inch.
3. Calculate the pitch and diameter of rivets for a cylindric vessel to withstand internal hydraulic pressure, the diameter of the vessel being 15 inches, plates $\frac{3}{4}$ inch thick of steel, with steel rivets, holes

drilled, double-riveted butt-joints with *one* cover plate; using strength rule for boilers, and taking 3 as a factor of safety. Calculate the greatest safe pressure of water inside the vessel.

4. Taking it that a steel punch may be worked up to 100,000 lb. per square inch, find the least diameter of hole that can be punched in the plates of Examples 2 and 3, the ultimate shear stress of the plates being taken respectively at 40,000 and 52,000 lb. per square inch.

5. A cylindrical boiler is to be constructed, 5 feet in diameter, to bear up to 400 lb. per square inch bursting pressure, the plates being iron, with iron rivets — holes punched, double-riveted butt-joints with two cover plates. Find the thickness of the plates. Ultimate stress as in Example 2.

NOTE.—In a question of this kind, find *first* the diameter d of the rivets by rule (4) on p. 128, then the thickness of the plates from the rule, $t = \frac{d^2}{1 \cdot 44}$.

ANSWERS.

1. $d =$ ·848 inch; pitch, 2·538 inches. Greatest safe pressure, 151·9 lb. per square inch.
2. $d =$ ·84 inch; pitch, 2·05 inches. Greatest safe pressure, 82·3 lb. per square inch.
3. $d =$ 1·04 inch; pitch, 2·73 inches. Greatest safe pressure, 433·8 lb. per square inch.
4. Iron plates, ·8 inch. Steel plates, 1·5 inch.
5. Diameter of rivets, ·707 inch. Thickness of plates, ·347 inch.

LESSON XXVI.

STRENGTH OF STRUTS.

THE practical rule for the strength of struts is based on Professor Gordon's formula, and is as follows:—

For a strut hinged at the ends the breaking load W

in pounds is obtained from the rule $W = \frac{A f}{1 + n B}$. Where A is the area of cross-section in square inches, f the breaking compressive stress of the material in lb. per square inch, n and B are constants depending on the dimensions and form of the strut. In these Examples f is taken at 80,000 lb. per square inch for cast iron, and 36,000 lb. per square inch for wrought iron.

The following short table gives values of n and B required in the Examples. A more extended table is given in Perry's "Practical Mechanics."

TABLE I.

Length of strut divided by least lateral dimension d.	B for cast iron.	B for wrought iron.
15	1·68	0·300
20	3·00	0·532
25	4·64	0·832
30	6·76	1·200
35	9·20	1·632
40	12·00	2·132

TABLE II.*

Shape of section. Least lateral dimension called d.	Value of n.
Square side d, or rectangle with smallest side d	1·00
Hollow rectangle least side d	0·50
Circle, diameter d	1·33
Circular, thin ring, external diameter d	0·66
Angle-iron, breadth of smallest flange d	2·00

For a strut *fixed* at the ends calculate by the rule given above, but take n one-fourth of the value given in the table.

Numerical Examples.

1. A hollow cylindrical column of cast iron is 15 feet long and 9 inches in outside diameter, the thickness of metal being $\frac{3}{4}$ inch. Find its breaking load, its ends being fixed.
2. Suppose a column like the last, of the same length and outside diameter, is required to bear a load of 50 tons. Find the proper thickness of metal, using 6 as a factor of safety.

* These two tables are quite distinct, the first giving the value of B for a particular ratio of length to least breadth, the second the value of n for a particular shape of cross section.

3. An angle-iron, used as a strut, is hinged at the ends, its length being 10 feet, least breadth 5 inches, and the area of its cross-section 15 square inches. Find its breaking load, and also its greatest safe load, using 6 as a factor of safety.
4. An angle-iron has to withstand a push of 10 tons. If its length is 10 feet and least breadth $3\frac{1}{2}$ inches—it being riveted firmly at the ends—find its proper cross-sectional area, using 6 as a factor of safety.
5. A cast-iron strut is 10 feet long and 4 inches square, *fixed* at the ends. Find its breaking load, 1st, by the rules given above; 2nd, by Euler's rule; and 3rd, by the formula worked out by Professor Perry (and given in the *Engineer* of December 10th and 24th, 1886), assuming in that case the error c to be 0·1. Breaking stress of the material, 80,000 lb. per square inch.

NOTE.—Professor Perry's formula is:—

$$2\,w = \beta\,m + f + \beta - \sqrt{(\beta\,m + f + \beta)^2 - 4\,\beta\,f}.$$

Where w is the breaking stress required.

β " " according to Euler $= \dfrac{E\,k^2\,\pi^2}{4\,l^2}$

f " " of the material.

(E = 17,000,000 for cast iron.)

$l = \frac{1}{4}$ length when *fixed*.

$= \frac{1}{2}$ length when hinged.

m a constant $= \dfrac{c \cdot o\,\beta}{k^2}$, where c is the given error in loading, etc., $o\,\beta$ the greatest distance from the neutral line, and k the radius of gyration of the section, *all dimensions being in inches.*

6. A wrought-iron strut is of the double **T**-section usual in rolled beams, breadth of flanges 5 inches, and total depth 9 inches, the mean thickness of metal everywhere being $\frac{3}{4}$ inch. If the strut is 15 feet long and *fixed* at the ends, find the height of brick wall, 14 inches thick and 10 feet long, it will carry safely, using 6 as a factor of safety. The section may be taken as a hollow rectangle, the two smallest sides being 5 inches broad and $\frac{3}{4}$ inch thick, the other sides being $\frac{3}{8}$ inch thick.

7. A strut of similar section to last, but 8 inches deep and 1 inch thick everywhere, is 30 feet long, and hinged at the ends. Find its *breaking* load by Euler's rule, E being 29×10^6.

ANSWERS.

1. 1,040,200 lb., or 464·37 tons.
2. Thickness of metal, $\frac{1}{2}$ an inch.
3. Breaking load, 90·49 tons. Greatest safe load, 15·08 tons.
4. Area, 6·8 square inches.
5. Breaking load by rules in text = 475,840 lb. Breaking load according to Euler = 994,130 lb. ($\beta = 62{,}133$). Breaking load by Professor Perry's rule = 375,376 lb. (w = 23,461).
6. Safe load, 65,407 lb. Height of wall, 50 feet.
7. 47,000 lb. (I about axis through centre of web = 21·33.)

LESSON XXVII.

GRAPHICAL STATICS, ETC. THE COMPOSITION OF FORCES ACTING *NOT* THROUGH ONE POINT. INTRODUCTION TO GRAPHIC METHODS. FORCES ON FRAMED STRUCTURES.

When forces do *not* act through one point they may no longer be regarded as of the same simple order of vector quantities, and hence a more complicated construction, or calculation, is required to obtain their resultant. For practical purposes *graphic* methods are usually preferable; and I shall now give a short introduction which, if the student will follow carefully, will enable him to master the subject sufficiently to attack any ordinary problem. Space only permits reference to forces acting in one plane. The conditions of equilibrium for such forces are, in the language of "graphics": (1) *The force-polygon must be closed*; (2) *The link-polygon must be closed.* The force-polygon is already familiar to the reader, and the link-polygon can best be described by working out one example.

In Fig. 61 four forces, A B, B C, C D, and D E, are shown which do not act through one point; Fig. 62 shows the force-polygon, A B C D E A, corresponding

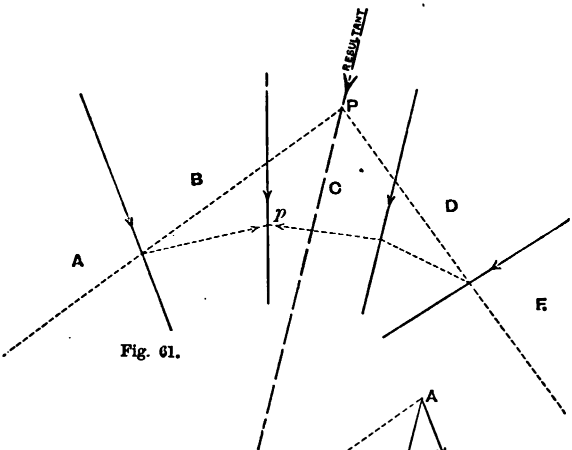


Fig. 61.

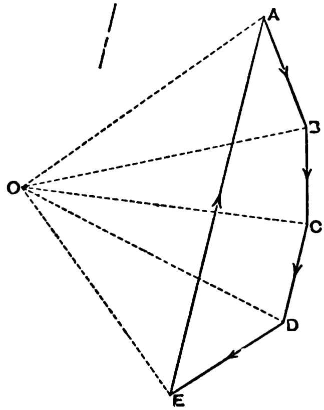


Fig. 62.

to them. E A gives the magnitude, direction, and sense of the equilibrant of the forces. The question now arises, where does the equilibrant act? To answer this question,

draw the link-polygon ; choose any point or pole—O in Fig. 62—and join this pole to each corner of the force-polygon, as shown in dotted lines.

Before going any further, notice how the figures are lettered. Each *space* in Fig. 61 has a letter assigned to it, and we speak of the force A B, meaning the force acting along the line which separates the space A from the space B. In Fig. 62 the force is represented by a line A B, each letter now standing at an *apex* of the polygon. To proceed, choose *any* point on one force, say A B (Fig. 61), and from it, through the space A, draw a line parallel to O A in Fig. 62; through space B a line parallel to O B, meeting the last on A B, and so on, till each space has its dotted line parallel to a dotted line in Fig. 62. The figure last drawn in 61 is called the *link-polygon,* and, if the forces are to be in equilibrium (with the help of the force we are seeking they will be), it must be closed; hence *close* it by producing the lines in A and E till they meet at P, the resultant must act through the point P thus found. Hence we have only to impose a force whose magnitude and direction are represented by A E in the force-polygon, at the point P, and our work is complete. The student should follow this work carefully; for if what we have now done is thoroughly understood, no great difficulty will be experienced with other and more complicated exercises. The following statements should be carefully noted and put to the test by the student :—

(1) It does not matter *what* point on A is chosen to begin the link-polygon.

(2) It does not matter what point in the plane of the forces is chosen for the pole O, except that if chosen in certain positions the drawing is of awkward dimensions.

The result will be the same if different positions of these two points are tried; and a point (not necessarily the same point) on the resultant will be obtained in each case.

Working out one or two examples, with drawing instruments, will do more to enable the student to understand this beautiful subject—which I have only

space to *introduce*—than any amount of reading. It is very much used now, especially in connection with the determination of the forces acting on the different pieces of structures such as railway girders and roof-trusses. In the few lines at my disposal I will endeavour to show how to determine (1) the supporting forces and (2) the "stress," or, more properly, the longitudinal *force* acting on each piece of such a structure, the loads and method of support being given. I must make the assumption that the pieces of the structure are fastened together with infinitely well-oiled pins, which ensures that the force

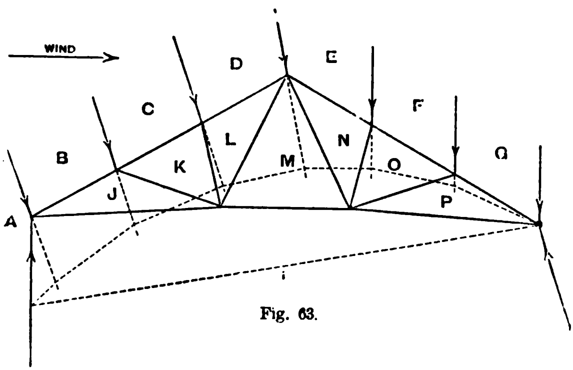


Fig. 63.

acting on each piece can act only along *its length*. First of all, however, about the supporting forces. It is usual to assume that one end is hinged, and the other supported on rollers to allow for expansion due to heating; if these rollers move freely, the force at that end will act vertically, the rollers moving on a horizontal surface. The supporting force at the hinged end (H, Fig. 63) may act in an inclined direction which is not at present known; all we know is *one point* in its direction, viz., the *hinge*. Having found the resultant load on each "bay" of one side of the truss, by combining load due to weights of parts, possible snow, etc., with wind-pressure, by the parallelogram of forces, and for the other

side taking simply the forces due to weights (since wind cannot act on both sides at once), our loads are now supposed found, and the "graphic" work proper begins. Draw the force-polygon A B C D E F G H (Fig. 64); this polygon is not yet complete, as we do not know the force acting at the hinge H. We *do* know the *direction* of the supporting force at the other end A; hence draw a vertical line of indefinite length from A in the force-polygon. Choose the pole O, and draw the radiating lines O A, O B, etc., as before; then draw the link-polygon (Fig. 63), *commencing at the hinge.* The forces A B, B C, etc., have to be produced downwards, as shown dotted, and the corners of the link-polygon rest on these lines as explained in the last example. Having completed the link-polygon as far as the supporting force at A, the polygon is now made to close by a line through space I. In the force-polygon (Fig. 64) draw from O a line parallel to this closing side; this line cuts the vertical line A I in the point I, which is the last corner, or apex, of the force-polygon. The polygon can now be completed, and the supporting force at H is represented by the line I H to the *same scale* to which A B represents its particular force or load. The student should very carefully work out for himself such an example as this, and by so doing he will learn more than is possible by mere reading.

I have not space to go very fully into the method of finding the "stress" figure for this truss, but will endeavour to put the student on the right track, so that he may complete the work without much difficulty. The force in each piece is supposed to act *along* the piece, so that at a corner, say A, we have four forces acting—the load A B, the supporting force A I, and the force due to the push or pull of each of the pieces B J and J I. If these forces are in equilibrium, they ought to be parallel and proportional to the sides of a polygon. Let us draw such a polygon. Well, in Fig. 64 we have already two of these sides, A B and A I; hence it is only necessary to complete the polygon, which is done by drawing a line from B parallel to B J (Fig. 63) and a line from I

parallel to I J. These lines intersect in the point J, and give us the polygon of forces, or "stresses," A B J I A for the point A. The line B J shows (to the same scale as that to which A B represents load) the force acting on the piece B J of the truss, trying either to crush it or to pull it asunder. Which of these does the force try to

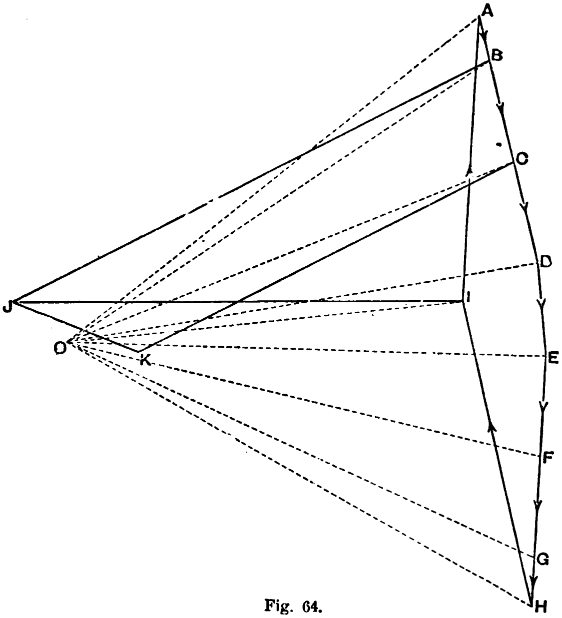


Fig. 64.

do? Well, this is easily found from the polygon A B J I; for the directions of the arrowheads on A B and A I are fixed, and the other arrowheads must be concurrent with these. Hence we see that the piece B J *pushes* the point A, and that the piece I J *pulls* the same point; or, in other words, the former is a *strut* and the latter a *tie*. Remembering that a piece which pushes with a given force at one end pushes *with an equal force*

at the other end, we know the action of the piece B J at the next point B C K J, and we can proceed to draw the "stress" polygon for it, as in the case of the last point. Two of the sides of the polygon we already have in Fig. 64, viz., B C and B J. The polygon when completed is found to be B J K C B; and by a similar process to that adopted before, the *character* of the force —whether tensile or compressive—can be determined.

Notice that each point where lines meet in the truss corresponds to a polygon in the stress figure, and that each line in the former is parallel to a corresponding line in the latter. Such figures are called *reciprocal figures.*

I have not *proved* to you that when the force- and link-polygons are closed, the system of forces is in equilibrium. The proposition in regard to the link-polygon is proved, by Professor Perry, in the following way :—Along each line, such as B, of the link-polygon (Fig. 61) introduce equal and opposite forces. This will not alter the equilibrium or want of equilibrium of the original forces. Now consider any corner, say *p*, of the link-polygon. The three forces acting there are parallel to the sides of the triangle B C O (Fig. 62), and hence are in equilibrium. In the same way it will be seen that the forces at each corner are parallel to the sides of a triangle in Fig. 62, and therefore in equilibrium. Hence, the whole system, consisting of the original and the superimposed forces, is in equilibrium. The superimposed forces are in equilibrium balancing in pairs; hence the *original forces* must have been in equilibrium. The student is now in a position to work out any of the examples which follow.

LESSON XXVIII.

EXPLANATION OF ANALYTIC METHOD OF SOLUTION.

HAVING found the supporting forces as explained at page 25, equate the horizontal components of all the forces acting at a joint, and also equate their vertical

components, the force in each piece being supposed to act longitudinally. This will give two equations, from which two unknown forces can be found. By properly choosing the sequence of joints all the forces can thus be found.

Numerical Examples.

1. A Warren girder is of the form and supports the loads represented in Fig. 65. Find graphically the

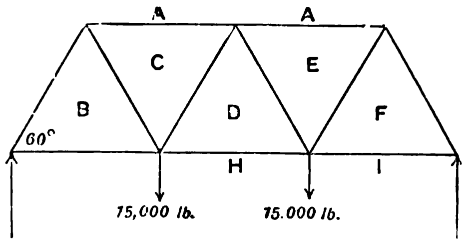


Fig 65.

longitudinal force which each piece withstands due to the given loads, specifying the character of the force—whether tensile or compressive.

2. A roof-truss, of the form shown in Fig. 66, is 30

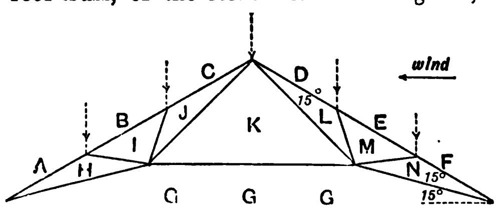


Fig. 66.

feet in span and supports the following loads:—viz., 1st, a dead load amounting to 30 lb. per square foot of horizontal projection of roof; and 2nd, a wind pressure on the right-hand side of 40 lb. per square foot normal to the roof. If the trusses are 8 feet apart, and the left-hand end is supposed to be hinged, the other end being supported on rollers,

find the tensile or compressive force which each piece of the truss has to withstand due to these loads. Also find these forces if the *other* end is taken as the hinge.

3. A horizontal prismatic beam is built firmly into two walls 20 feet apart and loaded with a uniform load of 200 lb. per foot, and also with loads of $\frac{1}{2}$ a ton at the middle and 5 feet from each end. Draw the diagram of bending moment, if the section is uniform. If the section is not uniform, but the depth given,

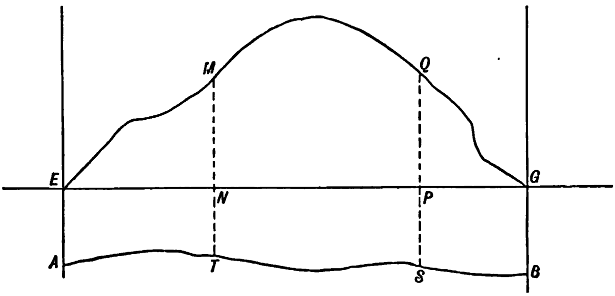


Fig. 67.

find the section everywhere if the beam is to be of uniform strength, and also in this case draw the diagram of bending moment.

The following general graphic solution of this question* is given in a paper by Professors Ayrton and Perry, read before the Physical Society on June 11th, 1887. In Fig. 67 the part E A B G E—this would be a rectangle in the first part of the question—has ordinates which represent the values of $\frac{1}{d}$ (the reciprocal of the depth of the beam), E M Q G E being a diagram whose ordinates represent m, the bending moment, if the beam were loaded as it is but merely supported at the ends. † Find, by trial,

* Accurate only for symmetrical loading.

† This is really the ordinary link-polygon for the forces acting on the beam ; the ordinate at any point of such polygon representing the bending moment at the corresponding point of the beam.

two points N and P, such that the area of E T + the area of P B = the area of N S, and M N = Q P. N and P, when found, represent points of inflection, and the real bending moment at any place is m − P Q, whence, knowing the bending moment everywhere and the depth d, it is easy to find the moment of inertia I, and consequently the section, as the condition of uniform strength is $I = \pm \frac{M d}{2 f}$, f being the stress to which the material may be subjected. The student will find no difficulty in completing the question.

ANSWERS.

EXAMPLE 1.—RESULTS FOR WARREN GIRDER.

A B	+ 17,320 lb.	A F	+ 17,320 lb.
B G	− 8,660	F I	− 8,660
B C	− 17,320	E F	− 17,320
C D	0	D E	0
A C	+ 17,320	A E	+ 17,320
D H	− 17,320		

EXAMPLE 2.

Total Force, + push, − pull.	Hinge at left-hand end.	Hinge at right-hand end.
In A H	+ 14,875 lb.	+ 16,600 lb.
B I	+ 12,950	+ 14,750
C J	+ 13,625	+ 15,800
D L	+ 18,000	+ 20,000
E M	+ 14,900	+ 15,800
F N	+ 19,075	+ 20,800
G H	− 10,975	− 15,000
H I	+ 1,675	+ 2,850
I J	+ 1,625	+ 2,850
J K	− 6,625	− 9,450
K L	− 12,725	− 14,500
L M	+ 4,675	+ 6,450
M N	+ 4,580	+ 6,350
N G	− 17,050	− 21,500
K G	− 4,875	− 8,600

LESSON XXIX.

CYLINDRIC AND CONICAL SPIRAL SPRINGS.

THE general law for the elongation of a cylindric spiral spring, subjected to an axial load W, is—

$$x = \mathrm{W}\, l\, r^2 \left(\frac{\cos^2 a}{\mathrm{A}} + \frac{\sin^2 a}{\mathrm{B}}\right),$$

and the relative turning of the ends is—

$$\phi = l\, \mathrm{W}\, r \sin a \cos a \left(\frac{1}{\mathrm{A}} - \frac{1}{\mathrm{B}}\right),$$

where l is the length of wire in the spring, r the mean radius of the coils, a the angle which the centre line of the wire makes with a plane at right angles to the axis of the spring, A and B being constants expressing respectively the torsional and flexural rigidities of the wire of which the spring is composed. Values of A and B, for a few common sections, are given below :—

Shape of Section.	Value of A.	Value of B.
Circle	$\frac{\pi \mathrm{N} d^4}{32}$	$\frac{\pi \mathrm{E} d^4}{64}$
Ellipse, diameters D and d Major diameter parallel to axis of spring	$\frac{\pi \mathrm{N D}^3 d^3}{16(\mathrm{D}^2 + d^2)}$	$\frac{\pi \mathrm{E D} d^3}{64}$
Square, side S	$\cdot 14058 \mathrm{NS}^4$	$\frac{\mathrm{ES}^4}{12}$
Rectangle; breadth b, thickness t, side b parallel to axis	$\frac{\mathrm{N}}{3} \times \frac{b^3 t^3}{b^2 + t^2}$ If t is very small compared with b, $= \frac{\mathrm{N}}{3} \times b t^3$	$\frac{\mathrm{E} b t^3}{12}$

A short proof of this general law may be interesting.

If the wire were everywhere horizontal, it would be twisted by a couple whose moment is W r. But the wire is *not* horizontal, hence (Fig. 68) measure off D A

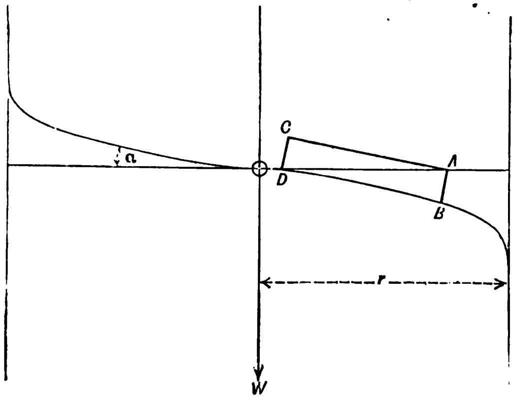


Fig. 68.

to represent W r, draw A B at right angles to the centre line O B, and complete the rectangle B C. Then W r cos a, the twisting moment (see page 4), is represented by D B, and W r sin a, the bending moment, by D C. Besides these there is a shearing force W distributed over the section, which, however, may be neglected. The angles through which unit length of spring moves, due to the twisting and bending moments respectively, are—

$$\frac{W\,r\cos a}{A} \text{ and } \frac{W\,r\sin a}{B}.$$

Projecting these angles on the horizontal plane, and paying proper regard to signs, we get—

$$\left(\frac{W\,r\cos a\sin a}{A} - \frac{W\,r\cos a\sin a}{B}\right) l = \phi \;.\;.\;. \;(1).$$

Next, project on the vertical plane, and

$$\frac{W\,r\cos^2 a}{A} + \frac{W\,r\sin^2 a}{B}$$

is the projection of the angular motion of unit length. Hence, the projection of length l being l times this, and that of the length of arc $l\,r$ times it, we have—

$$(2) \ldots \quad x = l\,r\left(\frac{W\,r\cos^2\alpha}{A} + \frac{W\,r\sin^2\alpha}{B}\right),$$

which agrees with the rules already given.

For a square section where α is so small that it may be taken as zero—

$$x = \frac{W\,l\,r^2}{\cdot 14058\,N\,s^4},$$

s being one side of the square.

For a spring of round wire, the angle of coiling, α, being very small, the rule for stiffness becomes—

$$x = \frac{32\,W\,l\,r^2}{\pi\,N\,d^4}.$$

The law for the circular section, last referred to, is easily worked out separately.

RULES FOR STRENGTH.

Circular section, $\alpha = 0$. The rule is—

$$W_s = \frac{\pi\,d^3}{16\,r}\,f_s.$$

For a square section, s being a side of the square, the maximum stress in the stuff due to the application of an axial load W is—

$$\frac{4{\cdot}79\,W\,r}{s^3}.$$

For a rectangular section $b > 6\,t$, it is—

$$\frac{3\,W\,r}{b\,t^2}.$$

The resilience of a spring of round wire in inch-pounds

$$= \frac{v}{4}\,\frac{f_s^2}{N}.$$

W_s is the greatest load it will bear without getting a permanent set, f_s the greatest elastic shear stress of the material, v being the volume in cubic inches of stuff

in the spring, and N the modulus of rigidity in lb. per square inch.

For a conical spiral spring—generally called a volute spring—such as that shown in Fig. 69, the rule for

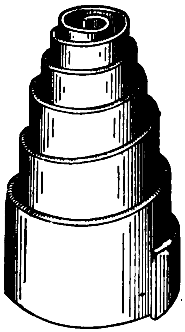


Fig. 69.

stiffness is—

$$x = \frac{3\,\pi\,W}{2\,\lambda\,N\,b\,t^3}\left(R_n^4 - R_o^4\right);*$$

where b and t are the breadth and thickness of the section respectively, λ the increment of radius per coil, R_n and R_o being the greatest and least mean radii of the spiral respectively. The greatest elastic load is—

$$W_s = \frac{b\,t^2}{3\,R_n}\,f_s.$$

Numerical Examples.

1. A spiral spring of the best tempered steel has a length of (round) wire 120 inches, mean radius of coils 1 inch, diameter of wire $\frac{1}{4}$ inch; how far will this spring elongate with a load of 50 lb.? Find the greatest lengthening before permanent set takes place, and the load producing this lengthening.

   $N = 13{,}000{,}000.$
   $f_s = 100{,}000$ lb. per square inch.

   Find also the resilience of the spring.

* "Minutes of Proceedings of the Institute of Civil Engineers," vol. ci., p. 265.

2. A spring is required—for a safety-valve—which is to shorten $\frac{1}{2}$ inch for a pressure on the valve of 100 lb. per square inch, the effective area of the valve being 12 square inches. The length of wire in the spring is 60 inches, the radius of the coils $1\frac{1}{2}$ inch; find the diameter of the wire, it being of best tempered steel. What force will just permanently injure this spring?
3. A cylindric spiral spring of round wire is required, which is to shorten $\frac{1}{2}$ inch for a load of 100 lb. If the mean diameter of the coils is 3 inches, and the number of coils 10, find the diameter of the wire, N being 13,000,000. If f_s is 100,000 lb. per square inch, find the greatest load which will not permanently injure the spring. Find also its resilience.
4. A cylindrical spiral spring of square wire is required, which is to shorten 1 inch under a load of 960 lb. If there are 9 coils in the spring, which is 3 inches in mean diameter, find the section of the stuff, the angle of coiling being very small. $N=12,000,000$. Taking the same proof-stress as in the last Example, find the greatest load which the spring will bear without permanent set. Find also the proper size of section if the angle of coiling is 45°, in which case the length of wire is $\sqrt{2}$ times as great as before. $E = 36,000,000$.
5. A spring similar to the last is required, the coils of which are to close right up with a load of 2,000 lb. If the space between two successive coils, when the spring is unloaded, is equal to the side of the section of the wire, find the proper size of section.
6. In a volute buffer spring the breadth of the strip is 4·5 inches, mean thickness 0·3 inch, maximum and minimum radii of coils (which are supposed just to fit so that $\lambda = t$) 2·6 and 1·4 inches respectively. Taking $N = 13,000,000$, and $f_s = 100,000$ lb. per square inch, find the deflection of the spring for a load of 2 tons; and also the greatest load it will bear without permanent injury.

ANSWERS.

1. Elongation for 50 lb., 1·203 inch. Elongation for permanent set, 7·38 inches. Load producing this elongation, 306·8 lb. Resilience, 1,133 inch-lb.
2. Diameter of wire, ·709 inch. Proof elongation, 1·95 inch. Proof load, 4,679·9 lb.
3. Diameter of wire = ·426 inch. Greatest safe load = 1,019 lb. Resilience = 216 ft.-lb.
4. Side of square = ·57 inch. Maximum load = 2,577·5 lb If angle of coiling is 45°, S = ·58 inch.
5. S = ·49 inch.
6. Deflection = 1·865 inch. Maximum load = 5,192·3 lb.

LESSON XXX.

CARRIAGE SPRINGS.

(Compiled by Professor Perry.)

RULES (refer to Fig. 70, where δ = overlap, and hence $n\,\delta = l$) :—

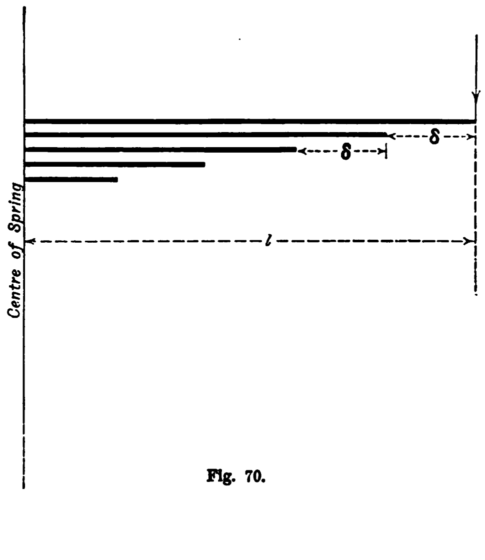


Fig. 70.

(1) $f = \frac{E\,t}{2}\,\omega$, (2) $f = \frac{6\,W\,l}{b\,t^2\,n}$, (3) $d = \frac{6\,W\,l^3}{n\,E\,b\,t^3}$.

Where t = thickness of each plate (in inches).
b = breadth " " "
d = deflection of spring "
n = number of plates.
E = Young's Modulus of Elasticity.
ω = change of curvature.
f = stress in plate due to bending.

N.B.—The spring is supposed to be just straight when subjected to its proof stress. It is easy to deduce the above rules from the laws of bending. The half-spring is like a cantilever, the bending moment on the top plate, say, increasing from the end to the point where it rests on the next plate, and then remaining constant = W δ, the maximum stress on all the plates being the same. Since

$$\omega = \frac{M}{E\,I} = 12\,\frac{W\,\delta}{E\,b\,t^3}, \text{ and } f = \frac{M\,y}{I};$$

also—

$$n\,\delta = l,\ \omega = \frac{2\,f}{E\,t}, \text{ and } f = \frac{6\,W\,l}{n\,b\,t^2},$$

as in (1) and (2), (3) is easily deduced.

Numerical Examples.

1. What ought the initial curvature of a $\frac{1}{2}$-inch plate to be, if the proof stress is 80,000 lb. per square inch? E = 30,000,000.
2. If the plate of (1) is 36 inches long ($2\,l = 36$), what is its initial dip?
3. If there are 6 such plates, what is the overlap of each on its neighbour?
4. If the breadth of each is 3 inches, find the deflection for a load (2 W) of 5,000 lb.
5. Find the load 2 W which will produce the deflection 1·72 inch—that is, which will just straighten the spring.
6. How many plates $\frac{3}{8}$-inch thick, 3 inches broad, the longest 30 inches long, of the above-mentioned steel will be required for a spring whose proof load is to be 2 tons?

NOTE.—In well-made carriage springs the resilience is $f_1^2 \div 6$ E inch-pounds per cubic inch, f_1 being the proof stress.

7. A spring is wanted of the above steel to take a proof load of 3 tons, with a deflection of 3 inches just making it straight. Find the total resilience, and hence the volume of the metal. If the total length ($2\,l$) is 36 inches, and b $3\frac{1}{2}$ inches, find t and n.

ANSWERS.

1. $\omega = \cdot 0106$, or radius of curvature $= 94$ inches.
2. 1·72 inch approximately. (Dip $= \dfrac{l^2}{2\,r}$ nearly).
3. Overlap, 3 inches.
4. $d = 1\cdot 296$ inch.
5. 6,635 lb. 6. 6 plates.
7. 10,080 inch-lb. Volume $= (n + 1)\, b\, t\, l$, initial dip $= 3$ inches; hence, if $2\,l = 36$, $b = 3\frac{1}{2}$, $n = 15$, and $t = \cdot 288$ inch.

LESSON XXXI.

SIMPLE HARMONIC MOTION.

RULES : $\dfrac{\text{Displacement}}{\text{Acceleration}} = \text{constant}$, and

$T = 2\,\pi\,\sqrt{\dfrac{\text{displacement}}{\text{acceleration}}}$, T being the periodic time of one complete vibration.

If the displacement from mean position be called x, then acceleration $= \dfrac{4\,\pi^2}{T^2}\,x$.

PROOF OF RULES.

Let the body moving on a circle have a constant angular velocity of A radians per second, and let time be reckoned from passing M (Fig. 71). The displacement

x of the projection agrees with the angular displacement A t of the radius.

Referring to the body moving on the diameter, with a motion which is the projection of the motion of the

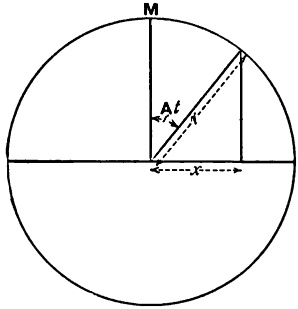


Fig. 71.

particle moving on the circle (the latter being uniform the former will be simple harmonic)—

$$\frac{d\,x}{d\,t} = \text{A}\,r \cos \text{A}\,t, \text{ since } x = r \sin \text{A}\,t.$$

$$\therefore \text{ acceleration} = \frac{d^2\,x}{d\,t^2} = -\,\text{A}^2\,r \sin \text{A}\,t,$$
$$= -\,\text{A}^2\,x.$$

Since T is the periodic time, $\text{A}\,\text{T} = 2\,\pi$,

$$\text{or } \text{A} = \frac{2\,\pi}{\text{T}},$$

and hence the rule becomes—

$$\text{acceleration} = -\left(\frac{2\,\pi}{\text{T}}\right)^2 x$$

$$= -\frac{4\,\pi^2}{\text{T}^2} \times x,$$

or neglecting the — sign

$$\text{T} = 2\,\pi\sqrt{\frac{\text{displacement } x}{\text{acceleration}}}.$$

The negative sign shows that the velocity *decreases* as the body moves away from its mean position.

It is easy from this to show that the periodic time of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum in feet.

Numerical Examples.

1. Find the length of a simple pendulum that will beat seconds at a place on the earth's surface where $g = 31{\cdot}29$.
2. In the College steam-engine the cross-head weighs 71 lb., piston 159 lb., piston-rod 32 lb., connecting-rod 114 lb.; the speed being 100 revolutions per minute. Find the accelerating force acting when the piston is at distances equal to $\frac{1}{8}$, $\frac{1}{4}$, $\frac{5}{12}$, and $\frac{23}{48}$ of the stroke from its middle position; crank, 1 foot; half the weight of the connecting-rod is neglected.
3. A U-tube consists of a semicircular part connecting two straight vertical parallel limbs. The distance between the straight parts is 18 inches, the internal diameter of the tube $1\frac{3}{4}$ inch (thickness of tube neglected). It is filled with water to a height of 9 inches in each straight part. If the water is displaced and then allowed to swing backwards and forwards, find the time of one complete vibration.

   NOTE.—It can be shown that the time of vibration is the same as that of a simple pendulum whose length is half the length of the liquid column.
4. A cage weighing one ton is supported by one mile of vertical steel wire rope one square inch in section. Assuming that 1 cubic inch of the rope weighs ·155 lb., E being 27,000,000, that the whole rope vibrates longitudinally with a simple harmonic motion, and that half the whole mass of the rope exists only at its lower end, the rest of its mass being neglected; find the periodic time of vibration.
5. A magnetic needle makes 20 complete vibrations in 3 seconds, and when 2 masses (each of 1 ounce) are placed each at 3 inches from the axis, and on opposite sides of it, it makes 20 vibrations in 4 seconds. Find the moment of inertia of the needle itself.

EXPLANATION.—Taking the rule $T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$, and remembering that angular acceleration $= \frac{\text{torque}}{\text{moment of inertia}}$, for the unloaded and loaded needle, with the same given displacement, the only thing in the right-hand expression which varies is the moment of inertia. Let displacement be called s and torque M.

$$\text{Then } \frac{T}{T_1} = \sqrt{\frac{\frac{M \div I_o}{s}}{M \div \left(I_o + \frac{2 W r^2}{g}\right)}} = \sqrt{\frac{I_o}{I_o + \frac{2 W r^2}{g}}}$$

$$\text{hence } \frac{T^2}{T_1^2} = \frac{I_o}{I_o + \frac{2 W r^2}{g}} \quad \text{or } I_o = \frac{2 W T^2 r^2}{g (T_1^2 - T^2)}.$$

I_o is the required moment of inertia of the needle itself, W the amount of each added weight in lb., r its distance from the axis in feet, T the time of vibration of the needle unloaded, and T_1 that of the needle when loaded.

6. Three equal similar and similarly supported discs vibrate in three different fluids. Their periodic times of vibration are approximately equal, being 3·5 seconds; the ratio of the amplitude of two successive swings, in one direction, being 0·98, 0·9, and 0·8 in the three cases respectively. Compare the viscosities of the fluids.

7. A pendulum consists of a spherical ball of wrought-iron 3 inches in diameter, supported by a wrought-iron wire $\frac{3}{8}$ inch in diameter and 16 feet long from the point of suspension (which is at one end) to the place where it enters the ball. Find the length of the simple pendulum which will vibrate in the same time.

ANSWERS.

1. 3·17 feet.
2. 271·6 lb., 543·2 lb., 905·34 lb., and 1,036·14 lb., respectively.
3. 1·569 second. 4. 1·32 seconds.
5. ·000312, one foot being taken as the unit of length.

6. Viscosities are as 90 : 45 : 8.
7. I = 47·57. Time of vibration, T = 4·107 seconds. Length of equivalent simple pendulum, 13·76 feet.

LESSON XXXII.

ANALOGIES BETWEEN THE LAWS OF LINEAR AND ANGULAR MOTIONS.

$\text{Acceleration} = \dfrac{\text{Force}}{\text{Mass}}.$	$\text{Angular acceleration} = \dfrac{\text{Moment or torque}}{\text{Moment of inertia}}$
1. A force of 200 lb. acts on a body which can move freely, its weight being 3,220 lb. Find its acceleration.	1*a*. A torque of 200 pound-feet acts on a fly-wheel with a very thin rim, the mean radius of the rim being 3·16 feet, and its weight 322 lb. Find the angular acceleration (friction being neglected). For I, see page 52.
2. A force of 50 lb. acts on a free body whose weight is 837·2 lb. Find the acceleration.	2*a*. A sphere of 2 feet diameter is composed of brass (which weighs 500 lb. per cubic foot), and can rotate freely about a diameter. If a torque of 50 pound-feet act on it, find the acceleration.
Velocity = acceleration × time.	Angular velocity = angular acceleration × time.
3. If the force mentioned in Ex. 1 act on the body for 6 seconds from rest, what will be the velocity of the body at the end of that time?	3*a*. If the torque mentioned in Ex. 1*a* act on the fly-wheel for 6 seconds from rest, what will be the angular velocity at the end of that time?
4. Find the linear acceleration of a sphere of cast-iron, 12 inches in diameter, which moves freely under the action of a force of 20 lb. If the sphere had already an acceleration of 4 feet per second per second along a line at right angles to the direction of the 20 lb. force, find the angular change in the direction of motion.	4*a*. Find the moment of inertia of a sphere of cast-iron of 12 inches diameter. If a couple whose moment is 20 pound-feet act on the sphere for three minutes from rest, find the angular velocity produced. If the body had already an angular velocity of 240 radians per second about an axis perpendicular to the first, find the angular change produced in the axis of rotation. (See page 54.)

Momentum = mass × velocity. 5. When the body mentioned in Ex. 1 is moving at the rate of 12 feet per second, find its momentum. If this momentum is destroyed in 5 seconds, find the average retarding force.	Moment of momentum = angular velocity × moment of inertia. 5*a*. When the fly-wheel mentioned in Ex. 1*a* is moving at 12 radians per second, find its moment of momentum. If this momentum is destroyed in 5 seconds, find the average moment stopping the wheel. If this moment consists of two equal and opposite tangential forces applied at opposite sides of the rim, find the amount of each force. If the coefficient of friction is ·1, find the amount of each pressure.
Distance passed through = $\frac{1}{2}$ acceleration × (time)^2. 6. How far will the body mentioned in Ex. 3 have moved at the end of the sixth second of its motion? 7. How far will the body mentioned in Ex. 2 have moved at the end of 5 seconds?	Angular distance passed through = $\frac{1}{2}$ angular acceleration × (time)^2. 6*a*. What angle will the wheel in Ex. 3*a* have described in the 6 seconds from rest? 7*a*. What angle will a radius of the sphere in Ex. 2*a* have described from starting till the end of the fifth second?
Energy = $\frac{1}{2}$ mass × (velocity)^2. 8. A body whose weight is 10,368·4 lb. moves with a velocity of 62·832 feet per second. Find its kinetic energy. 9. A body which weighs 322 lb. has a store of kinetic energy of 10,000 foot-pounds. What is its velocity?	Energy = $\frac{1}{2}$ moment of inertia × (angular velocity)^2. 8*a*. A fly-wheel whose moment of inertia is 322 revolves 10 times per second. Find its kinetic energy. 9*a*. A body whose moment of inertia is 10 possesses a kinetic store of 10,000 foot-pounds of energy. What is its angular velocity?
Horse-power = $\dfrac{\text{Force} \times \text{velocity in feet per minute}}{33{,}000}$ 10. The steady pull of the rope from an hydraulic capstan which draws a carriage is 100 lb., the carriage moving at a steady speed of 500 feet per minute. Find the HP exerted.	Horse-power = $\dfrac{\text{Moment} \times \text{angular velocity in radians per minute}}{33{,}000}$ 10*a*. A shaft is acted on by a couple whose moment is 100 pound-feet, the velocity of the shaft being 79·3 revolutions per minute. Find the HP given to the shaft.

Additional General Examples.

11. What moment applied to a wheel, whose rim weighs 5 tons, mean radius 3 feet, will get up a speed of 20 revolutions per second in one minute—friction being neglected?
12. A torque of 500 pound-feet is applied to the wheel in the last example; what is the angular acceleration produced?
13. What is the angular velocity of the wheel, in that case, at the end of 7 seconds from rest?
14. Through what total angle will the wheel have turned during the 7 seconds?

ANSWERS.

1. 2 feet per second per second.	1*a*. 2 radians per second per second.
2. 1·923 foot per second per second.	2*a*. 1·923 radian per second per second.
3. 12 feet per second.	3*a*. 12 radians per second.
4. 2·62 feet per second per second. Change of direction, 34° 7′.	4*a*. I = ·76 Angular velocity = 4731·5 radians per second. Angular change of axis, 87° 13′.
5. Momentum, 1,200. Average force, 240 lb.	5*a*. Moment of momentum, 1,200. Average moment, 240 lb.-feet. Each force is 37·9 lb. Each pressure is 379 lb.
6. 36 feet.	6*a*. 36 radians.
7. 24·037 feet.	7*a*. 24·037 radians.
8. 635,605·5 ft.-lb.	8*a*. 635,610 ft.-lb.
9. 44·7 feet per second.	9*a*. 44·7 radians per second.
10. 1·51 HP.	10*a*. 1·51 HP.
11. 6,556 lb.-ft. approximately.	12. ·16 radians per sec. per sec.
13. 1·12 radian per second.	14. 3·92 radians.

LESSON XXXIII.

HYDRAULICS. HYDRAULIC MACHINERY, ETC.

IN problems connected with the steady flow of water from one place to another, the fundamental law—

$$h + \frac{v^2}{2g} + \frac{f}{w} = \text{constant, for every pound of water,}$$

is of the greatest service to us. In this formula h is the height above datum in feet, v the velocity in feet per second, f the pressure in lb. per square foot, and w the weight of one cubic foot of the water. The law may be proved as follows:—Imagine a very small mass of water flowing along stream lines, as shown in Fig. 72. Imagine

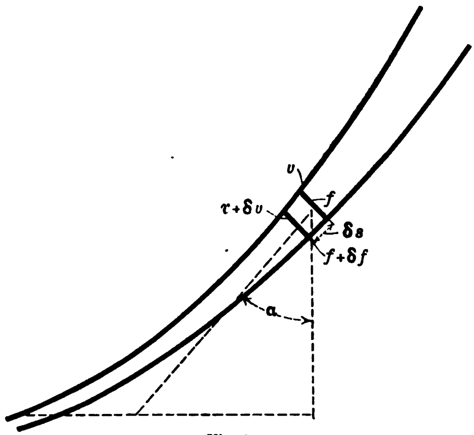


Fig. 72.

it to be a frictionless fluid acted on only by gravity. Let a be the cross-sectional area of the little column, its length being δs, the velocity and pressure being v and f at one end, $v + \delta v$ and $f + \delta f$ at the other. Since force = mass × acceleration, the resultant force in the direction of the stream tube is $\frac{w}{g} a \, \delta s \, \frac{\delta v}{\delta t}$.

Since the force of gravity resolved along the stream tube, together with the resultant of the pressures on the ends of the column, is equal to the acting force, it must equal the above expression.

The resolved part of gravity is $w\ a\ \delta\ s \cos a$, the resultant impressed force in the same direction is $f\ a - (f + \delta f)\ a$, hence—

$$\frac{w}{g}\ a\ \delta\ s\ \frac{\delta\ v}{\delta\ t} = w\ a\ \delta\ s \cos a + f\ a - (f + \delta f)\ a.$$

Dividing across by a we get—

$$\frac{w}{g}\ \delta\ s\ \frac{\delta\ v}{\delta\ t} = w\ \delta\ s \cos a - \delta f.$$

We have taken $\delta\ s$ any small element of length; take it such that $\frac{\delta\ s}{\delta\ t} = v$, also let $\delta\ s \cos a = -\ \delta\ h$; then our equation becomes—

$$\frac{w}{g}\ v\ \delta\ v = -\ w\ \delta\ h - \delta f,$$

$$\text{or } \frac{1}{g}\ v\ \delta\ v + \delta\ h + \frac{\delta f}{w} = 0.$$

Letting the quantities become indefinitely small, and integrating, we get—

$$\frac{v^2}{2\ g} + h + \int \frac{\delta f}{w} = \text{constant} \ldots (1).$$

Or in the case of water—

$$\frac{v^2}{2\ g} + h + \frac{f}{w} = \text{constant} \ldots (2).$$

These terms may be called respectively the kinetic, the potential, and the pressure energy of the 1 lb. of water. $2{\cdot}3\ p$ may be written for $\frac{f}{w}$ where p is the pressure in lb. per square inch. This law is often required in solving Examples, such as Nos. 11, 18, etc, and for the calculation of power wasted in hydraulic mains.

Numerical Examples.

1. To punch a certain hole in a boiler-plate requires the exertion of a force of 56,000 lb. This is produced

by the action of water, at a pressure of 750 lb. per square inch, on the ram of an hydraulic punching machine. What is the area, and what the diameter of the ram?

2. Taking the rule that the force required to punch a hole of diameter D in a plate of thickness T is 80 D T tons, what size of hole will the machine in Ex. 1 punch in $\frac{3}{8}$-inch plate?

3. A ram of 15 inches diameter can rise and fall in a press through a height of 12 feet. If the pressure of the water is 1,500 lb. per square inch, what weight may be placed on the ram? When this weight is at its full height it possesses a store of energy; how much is this store? Suppose this energy all given out in 15 seconds, what HP will it develop in hydraulic machinery of 40 per cent. efficiency (the weight of the ram, etc., being neglected)?

4. In a direct-acting hydraulic hoist the weight of the cage and ram is 2 tons, and the weight in the cage $\frac{1}{2}$ ton. The ram is 5 inches diameter, and 44 feet of it are in the press at the beginning. What ought to be the height of water in a tank communicating with the press to just work the hoist in its lowest position? What ought to be the height of the water in the tank when the hoist has risen 20 feet? What when it has risen 40 feet?

NOTE.—The pressure of the water in lb. per square inch, due to a "head" of h feet, is $\frac{h}{2 \cdot 3}$. The ram, when immersed in the water in the press, seems to lose weight; this loss being equal to the weight of the water displaced by it.

5. The fixed weight of the above-mentioned hoist (2 tons) is balanced by means of a chain and counterweight; find the weight of the chain per foot for perfect balance in all positions of the hoist, the pressure inside the press being supposed constant.

6. A water-pressure engine with three cylinders is worked by water from an accumulator at a pressure

of 700 lb. per square inch. If the diameter of each piston is 5 inches, stroke 1 foot, speed 55 revolutions per minute, find the indicated HP of the engine (it is single-acting).

7. In an Armstrong hydraulic crane the diameter of the ram is 9 inches, and the motion of the ram is magnified eight times by means of pulleys, so that the weight lifted moves eight times as fast as the ram. If this crane is driven by water at a pressure of 700 lb. per square inch, what will be its efficiency when lifting 2 tons?

   HINT.—Let the load W lb. be lifted 8 feet, the ram moving out 1 foot. W × 8 is the work given out by the machine, whilst the energy put into it is that contained in the water necessary to fill 1 foot of the space left by the ram. Every 1 lb. of the water has 2·3 × 700 ft.-lb. of energy in it, hence the energy in the water required to fill the 1 foot is easily found (the energy being mainly *pressure energy*). The ratio of W × 8 to this amount is the efficiency.

8. Suppose the crane in the previous Example, instead of having a ram of uniform diameter, has a piston of 9 inches' diameter with a piston-rod of 5 inches' diameter; if *both* sides of the piston are put in communication with the accumulator, what weight will the crane lift, the efficiency of the machinery being 62 per cent.?

9. What is the velocity with which water leaves a sharp-edged orifice in a vessel, 20 feet below the free surface of the water in the vessel? (Formula, $v^2 = 2\,g\,h$.)

10. A round orifice is 3 square inches in area, and the contracted vein is 0·6 of the area of the orifice; what is the rate of flow? Head as in last Example.

11. Twenty cubic feet of water pass per second along a pipe, keeping it full. At two places the sections are 3 square feet and 2 square feet respectively, difference of levels 10 feet. What is the difference of pressure assuming no friction?

12. Round a certain bend in a pipe of 2 square feet section (20 cubic feet per second being the flow) the

loss of energy due to friction is one-tenth of the kinetic energy of the water; what is the loss of pressure if there is no fall in the pipe?

13. D'Arcy's experiments tell us that each pound of water loses $f\frac{4\,L}{d}$ times its kinetic energy in flowing along a straight pipe of diameter d feet, length L feet, where f is ·0058 for a 6-inch pipe. A straight pipe 6 inches' diameter, 1,000 feet long, conveys 4 cubic feet of water per second; what is the total loss of energy per pound? Express it as a height.

14. A pipe which conveys water is 5 miles long and 6 inches in diameter; with 60 feet head, what will be the flow?

NOTE.—It is easy to deduce from D'Arcy's formula the rule $d^5 = \cdot 00058\,\frac{L\,Q^2}{H}$, d being the diameter, L the length of pipe, H the head, all in feet, and Q the quantity flowing in cubic feet per second. (See page 165 for variation of coefficient.)

15. A water main is to deliver 100,000 gallons per day of 24 hours; there are no bends in the pipe, which is 1 mile long and 2 feet in diameter. Find the loss of head due to friction in the pipe.

16. Petroleum oil is to be sent through a pipe a distance of 60 miles, the available head being 200 feet, and quantity to be sent 4 cubic feet per second. Find the proper diameter for the pipe on the assumption that the same rules hold as for water.

17. In the last Example, if the density of the oil be 0·88, what would you consider a proper thickness for the pipes, safe stress of the metal being taken at 2,600 lb. per square inch?

18. Find the total energy of 6 cubic feet of water flowing at the rate of 10 feet per second under a pressure of 500 lb. per square inch, and at a height of 30 feet above datum.

19. A cylindrical vessel 9 inches in internal diameter is filled to a height of 18 inches with water; it is

mounted on a raft, and placed in a still pond. There is a hole $\frac{3}{4}$ inch in diameter with sharp edges in the vessel, the centre of the hole being 16 inches from the surface of the water. Find the force acting on the vessel tending to move it along, the head of water inside being supposed constant, and weight of vessel and raft 6 lb. (Coefficient of contraction $= \cdot 6$.) Find also the acceleration of the vessel, neglecting the small resistance due to friction.

20. A centrifugal pump has a lift of 10 feet. If its outside diameter is 2 feet, find the speed of the pump, the vanes being radial to the outside circumference.

    NOTE.—The speed of the circumference should be that of a stone which has fallen freely through a height equal to half the head, friction being neglected.

21. A centrifugal pump delivers 80 cubic feet of water per minute. If the outside diameter is 3 feet, find the useful HP of the pump, the vanes being radial, speed 150 revolutions per minute, and friction neglected.
22. A centrifugal pump is going at 500 revolutions per minute, inside radius 1 foot, outside radius 2 feet; assuming that the delivery pipe is so high that the pump is delivering little water, find the change of pressure in lb. per square foot.
23. At what speed must the pump in Example 22 run, to produce a change of pressure = 15 feet of water?
24. An inward-flow turbine is required for a fall of 30 feet; find the speed at which it should run, the vanes being radial, and its outside diameter 3 feet.
25. A centrifugal pump is employed to lift water 10 feet. If the outside diameter of the revolving vanes is 2 feet, outside breadth 5 inches, and speed 171 revolutions per minute, find the quantity of water delivered in gallons per minute, friction being neglected. Find also the HP given to the water by the pump.

    NOTE.—The radial velocity of the water through the wheel is generally taken as $\frac{1}{8}$ of that due to the total fall.

If it is found in practice that it is necessary to run the pump at 250 revolutions per minute in order to get this flow, find the "hydraulic efficiency" of the pump.

The "hydraulic efficiency" is found by dividing the lift in feet by $\frac{v^2}{32 \cdot 2}$, where v is the velocity of the circumference in feet per second.

ANSWERS.

1. Area, 74·66 square inches. Diameter, 9·72 inches.
2. Diameter of hole, ·832 inch.
3. Weight, 118·34 tons. Energy, 3,180,840 ft.-lb. 154·22 HP.
4. 612·18 feet. 632·11 feet. 651·98 feet.
5. 4·247 lb. per foot. Weight of 2 feet of chain must = weight of 1 foot of water column displaced by ram.
6. 22·9 × 3 HP.
7. 80·46 per cent.
8. 1,065·3 lb.
9. 35·8 feet per second.
10. ·448 cubic foot per second.
11. Change of pressure, 3·968 lb. per square inch (the smaller section being the lower).
12. Loss of pressure, ·067 lb. per square inch.
13. Loss of energy, 298·8 ft.-lb. Loss of head, 298·8 feet.
14. ·349 cubic foot per second.
15. ·0033 ft.-lb. lost by each pound of water.
16. Diameter, 1·7 foot.
17. Thickness = ·3 inch.
18. 441,650 ft.-lb.
19. Force, ·305 lb. Acceleration, ·2 foot per second per second. (Force = momentum per second.)
20. Speed, 171·5 revolutions per minute.
21. 2·6 HP.
22. Change of pressure, 5,290 lb. per square foot (neglecting friction).
23. Speed, 171·4 revolutions per minute.
24. Speed should be 149 revolutions per minute. Velocity of circumference of wheel should be that of a stone falling freely through a height = half the head.
25. 3,736 gallons per minute. 11·24 HP. Hydraulic efficiency 47 per cent. nearly.

LESSON XXXIV.

HYDRAULIC AND ELECTRIC TRANSMISSION OF POWER.

In deducing an approximate formula for the power wasted in hydraulic mains, we may regard the whole energy of 1 lb. of the water, at a pressure of p lb. per square inch, as pressure energy, and hence equal to $2{\cdot}3\ p$ ft.-lb. If the flow is Q cubic feet of water per second, $p\ Q \times 2{\cdot}3 \times 62{\cdot}4$ or $144\ p\ Q$ ft.-lb. of energy enter the pipe per second, and therefore the HP entering the pipe (call it E) is—

$$= \frac{144 \times 60 \times p\,Q}{33{,}000} = {\cdot}2605\ p\,Q \ldots . (1).$$

But the loss per lb. is approximately $\cdot 0058 \times 4 \dfrac{L\,v^2}{d \times 2\,g}$ in L feet of straight pipe, and above we have $60 \times 62{\cdot}4$ Q lb., hence the loss of energy per minute is—

$$\cdot 0058 \times 4 \frac{L\,v^2}{d \times 2\,g} \times 60 \times 62{\cdot}4\ Q \text{ ft.-lb.};$$

and the HP wasted is—

$$\frac{\cdot 0058 \times 4 \times 60 \times 62{\cdot}4}{64{\cdot}4 \times 33{,}000} \cdot \frac{L\,v^2\,Q}{d} \ldots . (2).$$

To get all in terms of p, E, d and L, we have $\dfrac{E}{\cdot 2605\ p} = Q$ from (1); also we know that—

$$v = \frac{Q}{\frac{\pi}{4} d^2} \therefore v^2 = \frac{16\ Q^2}{\pi^2\ d^4} = \frac{16\ E^2}{(\cdot 2605)^2\,p^2\,\pi^2\,d^4},$$

or (3) HP wasted (call it W)

$$= \cdot 00374 \cdot \frac{L\,E^3}{p^3\,d^5}.$$

In these formulæ L and d are in feet, and p in lb. per square inch. It should be noted that the coefficient ·00374 is only strictly correct for a 6-inch pipe, but the variation in its value is not great for any likely value of

d, and it can be found from D'Arcy's coefficient, which is really $\cdot 005\ (1 + \frac{1}{12\ d})$.

If the HP *to be delivered* is given (call it D), then $E = D + W$, which must be substituted in the above rule (3); and W can be found by trial or by means of squared paper.

In electric transmission of power, if the conductor is given, the voltage and current at the sending station being P and C respectively, then $E = \frac{P \times C}{746}$, and the HP wasted in heating the conductor is $W = \frac{C^2\ R}{746}$, where R is the resistance of the conductor in ohms, and hence (4) $W = 746\ \frac{E^2}{P^2} \times R$.

The resistance of 1 *mile* of copper conductor 1 square inch in section, at the ordinary temperature, is ·04378 ohm; hence, if there are L *miles* of conductor altogether—

$$(5)\quad W = 746\ \frac{E^2}{P^2} \times \frac{L}{a} \times \cdot 04378\ \cdot = 32{\cdot}66\ \frac{E^2}{P^2} \times \frac{L}{a}.$$

If the current density is fixed at, say, 380 amperes per square inch—in other words, if each square inch of cross-section of conductor carries 380 amperes of current—the rule simplifies to

$$(6)\quad W = 16{\cdot}636\ \frac{E}{P} \times L.$$

Remember, L is in miles; and a, as the area of the cross-section of the copper conductor, in square inches.

ECONOMY IN CONDUCTORS.

If we wish the power wasted, by friction or resistance as the case may be, to be the smallest possible, consistent with greater cost of pipe, conductor, etc.,* it has been

* See an article on this subject by the Author, published in *Engineering* of May 22nd and June 5th, 1891.

shown that, for a pressure at entrance of 700 lb. per square inch, the diameter of the pipe should be—

$$d = {\cdot}079\ E^{\frac{3}{7}};$$

and for a pressure of 1,400 lb. per square inch, the rule is—

$$d = {\cdot}049\ E^{\frac{3}{7}}.$$

The similar electrical problem has been solved by several authorities. Lord Kelvin's rule is that the area of the conductor should be simply proportional to the *current*, or in other words the *current density*, a certain amount depending on the price of copper, the cost of 1 electrical HP, etc. A current density of about 380 amperes per square inch seems to be the best average. Professors Ayrton and Perry have also solved the problem * from the point of view of the power *delivered* at the distant station. Their rules are that the current should be—

$$C = \frac{w}{P}\,(1 + \sin\phi),$$

$$\text{and } r = \frac{P^2}{n\,w}\,\frac{\sin\phi}{(1 + \sin\phi)^2},$$

r being the resistance in ohms per mile, which should be given to the n miles of conductor, w the power delivered in watts, and ϕ is such an angle that $\tan\phi = \frac{n\,t}{P}$, t being a number (depending on cost of conductor, etc.), which may be taken as, or about, 17.

The above rules were deduced for continuous currents. For alternating currents there is little necessity for such calculations, the voltage being usually so high that the cost of conductor is of comparatively little importance.

The student will remember that in a circuit conveying alternating currents, virtual E.M.F. (as found, say, by a Cardew voltmeter) $\times \cos\phi$ = *effective* virtual E.M.F., and this $\div$ resistance gives virtual current. The product of

* See the *Electrician* for March, 1886.

E.M.F. and current, in volts and amperes respectively, thus obtained, must be multiplied by cos ϕ to get mean power developed in watts. ϕ is here the angle of *lag* which is such that $2 \pi n \mathrm{L} \div$ resistance in ohms is its tangent, L being the coefficient of self-induction, and n the number of periods per second.

Numerical Examples.

1. If 200 HP are sent into an hydraulic power main, 6 inches in diameter and 1 mile long, find the power wasted, the pressure being 700 lb. per square inch. If the pipe is 5 miles long, find the waste.
2. 200 HP are sent into a 6-inch pipe, the pressure being 1,400 lb. per square inch; find the power wasted by friction in 1 mile of pipe. If the pipe is 4 inches in diameter, what is the waste?
3. What is the greatest distance to which 500 HP can be transmitted through a 6-inch pipe with a waste of (1) 20 per cent., (2) 50 per cent.? Pressure as in Ex. 1.
4. 100 HP are to be delivered at a place one mile distant, the diameter of the pipe being 6 inches, and the pressure at entrance 700 lb. per square inch; find the power wasted. If the pipe were 4 inches in diameter, find the waste necessary, and hence the power which must be provided at the sending station.
5. If, in the case of the 4-inch pipe, 100 HP are merely *sent in*, what is the waste by friction in the pipe?
6. The current for an electric tramway is supplied from a generating station at one end of the line which is 5 miles long, there being a return conductor. If the pressure at entrance is 200 volts, and 100 HP are sent in, find the power wasted in transmission, and the power which arrives at the other end of the line. Current density, 380 amperes per square inch.
7. If it were necessary to *deliver* 100 HP at the distant end, what would be the power wasted, and hence what power must be sent in?
8. If we wish to transmit power electrically to a distance of 3 miles—there being a return conductor—the diameter of the copper wire conductor being 0·7

inch, the pressure at entrance 300 volts, and 50 HP being sent in, find the power wasted in heating the conductor.

9. In the 5 miles of tramway referred to in Example 6, if the diameter of the copper wire conductor is 1·128 inch, 100 HP being supplied at the generating station, and 30 HP required at the distant end, find the voltage necessary at the generating station.
10. Hydraulic system, 200 HP sent in, pressure 700 lb. per square inch, distance 5 miles. Electric system, same HP sent in, pressure 700 volts at entrance (in the two cases, the diameter of the pipe and conductor being respectively that most consistent with economy), find the waste of power.
11. If in the last case the voltage at entrance had been 2,000, what would have been the waste of power?
12. In question 10 what is the internal diameter of the hydraulic main?
13. In a circuit converging an alternating electric current the E.M.F. as measured by a Cardew voltmeter is 100 volts (virtual), the coefficient of self-induction is ·001 henry, the resistance 2 ohms, and the frequency 65 periods per second; find the current and the mean power.

ANSWERS.

1. 14·72 HP. 73.6 HP. 2. 1·84 HP. 14·48 HP.
3. 2,293 feet. 5,732 feet.
4. About 2 HP. 38·25 HP. 138·25 HP.
5. 14·48 HP. 6. 83·18 HP wasted. 16·82 HP arrive.
7. HP wasted, 64·75; hence 164·75 HP must be sent in. Area of conductor in this case found from Ayrton and Perry's rules. Best resistance, ·01279 ohm per mile.
8. 14·1 HP. 9. 216 volts.
10. Hydraulic, 8·6 HP. Electric, 47·5 HP.
11. 16·6 HP. 12. ·76 foot.
13. Current, 48·98 virtual amperes. Mean power, 4,802 watts.

LESSON XXXV.

THE STEAM ENGINE.

Numerical Examples.

1. If the calorific power of a certain kind of coal is 8,200 centigrade (British) thermal units per lb., what are the efficiencies of engines (and boilers) which produce 1 HP by the burning of $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, and $3\frac{1}{2}$ lb. of coal per hour?

   One (British) centigrade thermal unit is equivalent to about 1,400 ft.-lb., and one Fahrenheit thermal unit to about 778 ft.-lb. of mechanical energy. The British thermal unit is the amount of heat required to raise 1 lb. of water one degree in temperature. (See page 35.)

2. How much water, fed at 40° C., ought to be converted into steam at 101·9 lb. per square inch pressure by 1 lb. of the above coal, if the boiler were perfect? If the efficiency of the boiler is 0·6, what is the feed-water evaporated per lb. of coal?

   Note.—The following rule is due to Regnault: $H = 606{\cdot}5 + {\cdot}305\,t$; where H is the total heat of 1 lb. of steam at a temperature $t°$ C, that is, the heat required to raise 1 lb. of water from 0° C., and to evaporate it at $t°$ C.

   It will be found from a reference to tables given in any book on steam that a pressure of saturated steam of 101·9 lb. per square inch corresponds with a temperature of 165° C.

3. If the evaporation is not taken as the actual quantity of water evaporated, but as if water at 100° C. were merely changed into steam at 100° C., what would be the evaporative power of the boiler in the last example?

4. If the average coal supplied to engines may be taken as containing 7,200 heat units(C.), find the efficiencies of the following engines (including boilers, etc.) respectively, viz.:—

   1. A very good marine engine using 2 lb. of coal per actual HP per hour.
   2. A good factory engine using 3 lb. of coal per actual HP per hour.

3. The College engine using $3\frac{1}{2}$ lb. of coal per actual HP per hour.
4. A good non-condensing engine using 6 lb. of coal per actual HP per hour.
5. An ordinary small non-condensing engine using 12 lb. of coal per actual HP per hour.
6. The worst old-fashioned non-condensing engines using 20 lb. of coal per actual HP per hour.

5. An indicator diagram is divided horizontally into 18 equal parts by ordinates drawn at right angles to the atmospheric line, the lengths of these ordinates being as follows: 0, 1·56, 1·55, 1·38, 1·14, ·95, ·81, ·71, ·65, ·59, ·51, ·41, ·4, ·35, ·31, ·3, ·26, ·25, 0 inch respectively; the greatest length of the diagram parallel to the atmospheric line being 4·45 inches. Scale of pressure, 32 lb. per square inch to one inch. The engine from which the diagram was taken had the following dimensions: diameter of piston 12 inches, length of crank 12 inches, and the speed 98 revolutions per minute. Find the indicated HP of the engine, using Simpson's rule to find the mean pressure of the steam from the diagram. Simpson's rule is as follows:—

$$\text{Area} = \frac{x}{3}\left\{\begin{array}{c}\text{sum of end ordinates} + 4\ (\text{sum of even ordinates})\\ + 2\ (\text{sum of odd ordinates}),\end{array}\right\}$$

where x is the common distance apart of any two successive ordinates.

$$x \text{ in this case} = \frac{4{\cdot}45}{18} = {\cdot}246 \text{ inch.}$$

Hence area (in square inches) =

$$
{\cdot}082\left\{\begin{array}{l}(0 + 0) + 4\ (1{\cdot}56 + 1{\cdot}38 + {\cdot}95 + {\cdot}71 + {\cdot}59\\ + {\cdot}41 + {\cdot}35 + {\cdot}3 + {\cdot}25) + 2\ (1{\cdot}55 + 1{\cdot}14\\ \quad + {\cdot}81 + {\cdot}65 + {\cdot}51 + {\cdot}4 + {\cdot}31 + {\cdot}26)\end{array}\right\}
$$

$$\text{Area} = {\cdot}082\left\{0 + 26 + 11{\cdot}26\right\}$$

$$\text{Area} = 3{\cdot}055 \text{ square inches.}$$

$$\text{Hence mean height} = \frac{3{\cdot}055}{4{\cdot}45} = {\cdot}686 \text{ inch.}$$

Mean pressure $= {\cdot}686 \times 32 = 21{\cdot}95$ lb. per square inch; whence the indicated HP can be easily obtained by the usual rule.

6. The breadths of an indicator diagram taken from a steam engine are 1·52, 1·55, 1·58, 1·52, 1·42, 1·13, 0·94, 0·70, 0·62, and 0·55 inch, the scale being 32 lb. per square inch to the inch. The crank of the engine is 12 inches long, diameter of piston 17 inches, speed 110 revolutions per minute. Find the indicated HP. (Simpson's rule is not applicable to this case, as there is an *even* number of ordinates.)
7. If the area of an indicator diagram is 4·232 square inches, and its greatest length parallel to the atmospheric line is 3 inches, scale of pressure, dimensions, and speed as in Example 6, what is the indicated HP?
8. Suppose you are designing a steam engine, the law of the expansion of the steam in the cylinder of which will probably be $p\ v =$ constant; the cut-off being at $\frac{1}{10}$ of the stroke, the initial pressure being 60 lb. per square inch above atmosphere, the back pressure 3 lb. per square inch above vacuum; find from theoretical considerations—

   (*a*) The mean effective pressure in the cylinder.
   (*b*) The work done by a cubic foot of steam in the cylinder.
   (*c*) The work done by 1 lb. of steam.
   (*d*) The probable indicated HP.
   (*e*) The number of lb. of steam required per minute.

   Dimensions as in Example 5.

Explanation.—The following rules will be found in Dr. Rankine's "Steam-Engine" and Professor Perry's "Steam."

(*a*) Let p_e, p_1, p_3, be the mean effective pressure, initial pressure, and back pressure respectively, measured from vacuum. M a constant depending on the law of expansion. For the law $p\ v =$ constant, $M = \frac{1 + log_e\ r}{r}$, steam being cut off at $\frac{1}{r}$th of the stroke. These rules, and others used in this Example, are founded on the

supposition that the diagram consists of straight lines and a regular expansion curve, as shown in

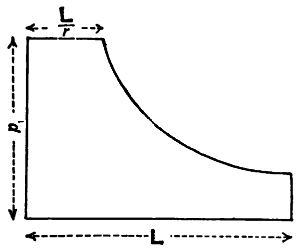


Fig. 73.

Fig. 73. In this particular case the curve is a rectangular hyperbola.

Then $p_e = p_1 \text{M} - p_3$, where

$$\text{M} = \frac{1 + \log_e 10}{10} = \cdot 3303.$$

Hence $p_e = 74{\cdot}7 \times \cdot 33 - 3 = 21{\cdot}65$ lb. per square inch. This is the answer to part (*a*).

(*b*) The work done by 1 cubic foot of the steam $= 144\, r \times p_e = 144 \times 10 \times 21{\cdot}65 = 31{,}176$ ft.-lb.

(*c*) The work done by 1 lb. of steam at an initial pressure $p_1 = \dfrac{\text{work done by 1 cubic foot of the steam}}{\text{D}}$.

Where D is the weight in lb. of 1 cubic foot, D may be found from a table given by Rankine, or it may be calculated from the empirical formula $\text{D} = \dfrac{p^{\frac{16}{17}}}{26{\cdot}36}$, where p is the pressure in atmospheres; whence $\text{D} = \cdot 1752$.

∴ Answer to part (*c*) is 177,945 ft.-lb.

(*d*) The probable indicated HP is easily found in the usual way, the mean effective pressure p_e being known. Answer, 28·5 HP.

(*e*) To find the number of pounds of steam required per

minute we have the rule: No. of pounds of steam required per minute $= \frac{\text{D A V}}{144\, r}$, where A is the area of the piston in square inches, and V its mean velocity in feet per minute. Answer, 5·28.

Or the question could be worked by simple proportion; for from part (*c*) we found that 1 lb. of steam does 177,945 ft.-lb. of work, and we want $28{\cdot}5 \times 33{,}000$ ft.-lb. done every minute; the question is how many lb. of steam are required every minute. Evidently $\frac{28{\cdot}5 \times 33{,}000}{177{,}945}$ lb. $=$ 5·28 lb. as before.

9. The law of expansion in a certain steam engine being $pl^{0.9} =$ constant, steam cut off at $\frac{1}{4}$ of the stroke, initial pressure 100 lb. per square inch, back pressure 3 lb. per square inch—
   (*a*) Find the work done in half a revolution of the crank-shaft, if the diameter of the piston is 12 inches, and length of crank 12 inches.
   (*b*) If the clearance at one end is ·02 of the volume of the cylinder, find the weight of steam used.
   (*c*) What is the work done per lb. of this steam?
   (*d*) What is the weight of steam used in the cylinder per indicated HP per hour? Speed, 100 revolutions per minute.
   (*e*) If the waste steam for the steam jacket is 10 per cent. of what is used in the cylinder, what weight of water is evaporated per indicated HP per hour?
   (*f*) If 1 lb. of coal evaporate 9 lb. of water under the actual conditions, what is the amount of coal used per hour per indicated HP?
   (*g*) What is the indicated HP?

   NOTE.—If the law of expansion is generally $pl^k =$ constant, then the quantity M mentioned in the last Example $= \frac{kr^{-1} - r^{-k}}{k-1}$.

10. In an indicator diagram the law of expansion is $pl^{0.85} =$ constant, steam is cut off at $\frac{1}{8}$ of the

stroke, and the initial pressure is 100 lb. per square inch. Find the rate per foot travel of the piston at which evaporation (or condensation) is taking place in the cylinder. Rate at which evaporation takes place in the cylinder is approximately $= \dfrac{1 - \frac{16}{17} k}{l}$ per foot travel of the piston, l being the distance passed through by the piston from the beginning of its stroke; this will give the evaporation as a fraction of the whole weight of steam already present in the cylinder. Answer, $\frac{0 \cdot 2}{l}$. (If the answer came out negative, it would indicate that *condensation* is taking place.)

11. The back pressure is $3\frac{1}{2}$ lb., and the cushion pressure 8 lb. per square inch; the clearance at one end of the cylinder is 0·025. What is the weight of steam used per stroke, the diameter of the piston being 15 inches, and the length of crank 13 inches? The engine makes 125 revolutions per minute. Find the indicated HP, and the weight of steam used per HP per hour.

12. Taking the heat given to a cubic foot of steam, as given by Rankine's formula, $H^* = 144 p_1 (M r + 15)$, find the efficiency of the cylinder. If in actual working it is found that 1 lb. of the coal used evaporates 6·25 lb. of water, find the consumption of coal per IHP per hour.

13. Accepting as true that the expansion curve of an indicator diagram follows the law $pl^k =$ constant,

(*a*) Find the law of the expansion curve of a diagram, the following being the values of p and l, represented by the ordinates of points on that curve.

(*b*) Make out a table showing the actual amount of evaporation taking place at different parts of the stroke. Plot a curve showing—to a scale comparable with the pressure scale—the evaporation

* H is in ft. lb.

taking place. (It will be condensation, or *negative* evaporation, if k is greater than $\frac{17}{16}$.)

No. of ordinate.	p in lb. per square inch.	l in inches.	No. of ordinate.	p in lb. per square inch.	l in inches.
1	45	7·09	5	30	12·39
2	41·25	8·282	6	26·25	14·309
3	37·5	9·887	7	23·125	17·635
4	33·75	10·873	8	19·175	22·309

(*c*) Find the law of this evaporation curve.

(*d*) Calculate the rate—per foot travel of the piston—at which the water-stuff is receiving heat (in ft.-lb.). This you can represent by a curve, the ordinates of which will be comparable with the pressure ordinates; since pressure, at any point on the diagram, is the *rate* per foot travel at which work is being done on the piston, the ordinate of this curve will represent, at every point, the *rate* at which the stuff receives heat. Let the ordinate of this curve be s times the corresponding pressure ordinate, then—

$$s = \frac{1}{f}\frac{d\,\mathrm{H}}{d\,v} = 1 + 29{\cdot}67\,f^{-{\cdot}1143}\,(1 - {\cdot}8857\,k)$$
$$- {\cdot}2762\,k f_1^{\,{\cdot}3303 - \frac{1}{k}} f^{\frac{1}{k} - {\cdot}818},$$

where f is the pressure in lb. per square foot at any point in the stroke and f_1 is the pressure (also in lb. per square foot) at the point where the quantity of water-stuff in the shape of steam is greatest. It should be noted in the above that changing the units in which p and l are measured does not affect the *index* in the law $pl^k =$ constant. It will also be readily understood that l may be inserted instead of v, since volume is simply proportional to length of cylinder occupied.

The following table gives values of *s*, which have been calculated for various values of *k*:—

Pressure in lb. per sq. inch.	Pressure in lb. per sq. foot.	Calculated Values of *s*.					
		$k = \cdot 8$.	$k = \cdot 9$.	$k = 1{\cdot}0$.	$k = 1{\cdot}111$.	$k = 1{\cdot}2$.	$k = 1{\cdot}3$.
8·4	1,206	4·41	3·09	1·86	·41	−·85	−2·29
14·7	2,116	3·94	2·82	1·69	·36	−·81	−2·14
39·3	5,652	3·22	2·32	1·40	·27	−·74	−1·88
69·2	9,966	2·72	2·01	1·23	·21	−·71	−1·74
101·9	14,680	2·36	1·79	1·10	·18	−·68	−1·65

From the above it is evident that if $k = 1{\cdot}13$ the expansion is adiabatic, and if k is greater than this the stuff is receiving *negative* amounts of heat; the ordinates of the curve would then be measured *downwards*. Rankine gives $k = 1{\cdot}111$ for adiabatic expansion, but this is incorrect; Zeuner gives 1·135.

14. In another indicator diagram the ordinates of the expansion curve are the following:—

No. of ordinate.	*p* in lb. per square inch.	*l* in inches.	No. of ordinate.	*p* in lb. per square inch.	*l* in inches.
1	47·7	5·9	7	30	10·95
2	44·7	6·411	8	27	12·766
3	41·7	6·978	9	24	14·751
4	39	7·744	10	21	17·815
5	36	8·595	11	18	22·411
6	33	9·702			

Work out the various items as stated in Example 13.

ZEUNER'S VALVE DIAGRAM (Fig. 74).

Draw the line A B to represent the line of centres of the engine, and C D at right angles to A B. Set off the angle C O E = the angle of advance, *against* the direction in which the crank moves. Set off

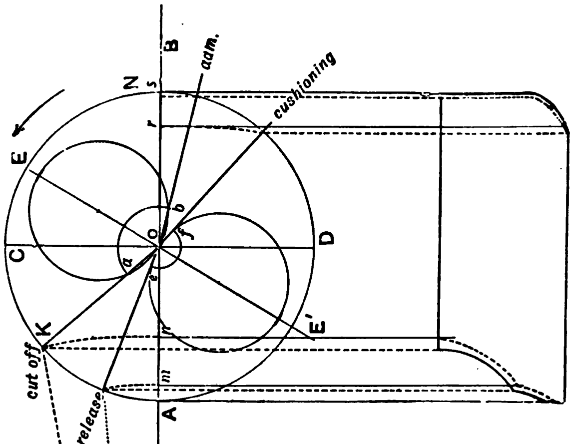


Fig. 74.

on O E and O E′ (*i.e.*, O E produced) distances each equal to the *half* throw of the eccentric. On these distances as diameters describe circles—called the valve circles. From O as centre with a radius = the outside lap cut the upper circle at *a* and *b* by an arc. Similarly describe an arc with radius = the inside lap cutting the lower circle in *e* and *f*. Join O *a*, O *b*, O *e*, O *f*. Then the main crank is parallel to O *b* when admission of steam takes place, is parallel to O *a* at cut-off, to O *e* at release or exhaust, and to O *f* when cushioning begins.

To get an idea of the points in the stroke of the piston at which cut-off, etc., take place, project down the points of intersection of the lines O *a*, O *b*, O *e*, and

O *f* with a circle of any convenient size drawn from O as centre to represent the path of the crank-pin; and draw the probable indicator diagram as shown in the sketch. In all this it has been assumed that the connecting-rod is infinitely long. To correct for the untruthfulness of this assumption, take a length G K to represent the length of the connecting-rod to the same scale that O K represents the length of the crank. From G as centre describe an arc through K, cutting O A in *n*; a similar arc with the same length as radius, cutting O A at *m*, etc. Project the new points *m*, *n*, *r*, *s*, to get the corrected diagram.

15. A certain link-motion gives the following values of travel and advance for different positions of the gear respectively. Work out by Zeuner's diagrams the hypothetical indicator diagram corresponding to each position of the gear, the outside lap being $\frac{3}{4}$ of an inch, inside lap $\frac{3}{8}$ of an inch.

Travel.	Advance.
$4\frac{1}{2}$ inches.	34°
$3\frac{1}{4}$ "	42°
$2\frac{1}{2}$ "	53°
2 "	70°
$1\frac{3}{4}$ "	90°

Calculate the fraction of the stroke performed before cut-off of steam occurs in each case.

ANSWERS AND SOLUTIONS.

THE STEAM-ENGINE.

1. Efficiencies, $11\frac{1}{2}$, 8·62, 6·9, 5·74, and 4·92 per cent.
2. 13·294 lb. of water, if boiler were perfect. 7·97 lb. of water in the second case.
3. Evaporative power, 15·3, if boiler perfect. 9·18, if boiler efficiency = 0·6.
4. In the first case, efficiency, ·098. Second, ·066. Third, ·056. Fourth, ·033. Fifth, ·016. Sixth, ·009 (less than 1 per cent.).
5. Indicated HP, 29·489.
6. Indicated HP, 111·6.
7. Indicated HP, 136·55.

9. (*a*) 13,345 ft.-lb. (*b*) Weight of steam used, ·1186 lb., calculating from pressure and volume just before opening of exhaust. (*c*) 112,521 ft.-lb. (*d*) 17·6 lb. (*e*) 19·36 lb (*f*) 2·15 lb. of coal per indicated HP. (*g*) 80·88 indicated HP.

11. ·204 lb. of steam required per stroke. Subtract the weight of steam filling the clearance space from the total weight of steam present at cut-off to get the weight of steam required per stroke. 202·49 indicated HP. 15·112 lb. of steam required per indicated HP per hour.

12. Efficiency of cylinder, 12 per cent. 2·42 lb. of coal required per indicated HP per hour.

13.* Plotting log. p and log. l as the co-ordinates of points, we get the curve shown in Fig. 75, from which the law is deduced as below:—

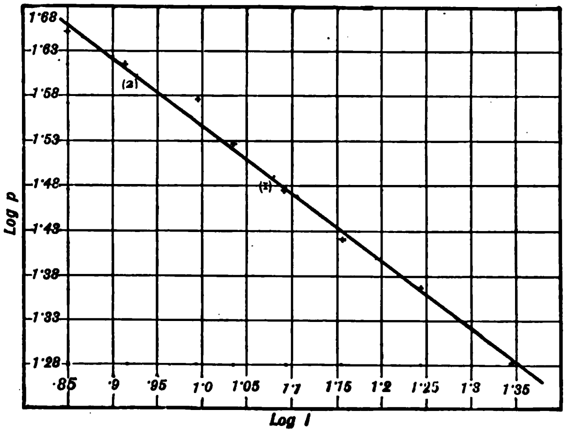


Fig. 75.

$$\text{Log. } p + k \log. l = \log. c.$$
$$(1)\ 1·49 + k\ 1·08 = \log. c.$$
$$(2)\ 1·607 + k\ ·93 = \log. c.$$
$$·15\ k = ·117.$$
$$k = \frac{117}{150} = ·78.$$
$$\text{Log. } c = 2·3324.$$

Law of expansion curve is $p\ l^{·78} = 215$.

* The author desires to express his indebtedness to Mr. W. C. Clinton, formerly an assistant in the Mechanical Engineering Department, for working out some of the results here given.

The student can, in this work, test the accuracy of the statement made at the end of page 175.

Table showing the amount of evaporation at different parts of the stroke:—

	p in lb. per sq. inch.	Vol. in cubic feet.	Weight of one cu. ft. of Steam.	Actual weight of Steam.
1	45·	·464	·1086	·05039 lb.
2	41·25	·5336	·10046	·053605
3	37·5	·64717	·0913	·05908
4	33·75	·71157	·0829	·058989
5	30	·81053	·0744	·060303
6	26·25	·93714	·06577	·061633
7	23·125	1·25406	·05794	·072656
8	19·175	1·46	·048	·07008

The dotted line shown (Fig. 76) is the evaporation curve as found, but as there are necessarily inaccuracies in the measurements of p and l, it is better to draw a regular curve.

Knowing k, we may find the law of the evaporation curve.

$$\frac{1}{m}\frac{dm}{dl}=\frac{1-\cdot 9393\,k}{l}. \quad k=\cdot 78, \text{ and } m = \text{mass of steam.}$$

$$\frac{1}{m}\frac{dm}{dl}=\frac{\cdot 26735}{l}.$$

$$\frac{dm}{m}=\cdot 26735\frac{dl}{l}.$$

$$\text{Log. } m=\cdot 26735 \text{ log. } l + c.$$

$$m = \mathrm{C}\, l^{\cdot 26735.} \quad \mathrm{C} = \text{antilog.} c.$$

14. Law of expansion curve is $p\, l^{\cdot 76} = 185\cdot 2$; the remaining parts of the Example can be worked out as in the solutions given above.

15. The steam is cut off at ·797, ·675, ·5, ·261, and ·071 of the stroke respectively.

LESSON XXXVI.

THE GAS ENGINE.

It has been found that one cubic foot of average coal-gas gives out about 530,000 ft.-lb. of energy in burning, and that 1 lb. of coal yields, on an average, about 4·5

cubic feet of coal-gas. The average coal used for gas-making gives out about 9,250,000 ft.-lb. of energy in

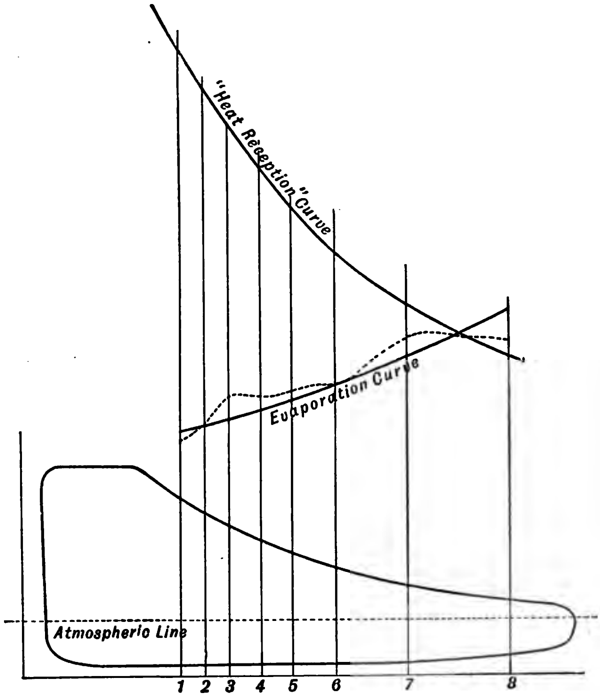


Fig. 76.

careful burning. Efficiency $= \frac{\text{energy given out}}{\text{energy supplied}}$, both in same units.

Numerical Examples.

1. The Lenoir gas engine required 95 cubic feet of coal-gas per hour per indicated HP; what was its efficiency as a heat engine? What is the efficiency of the whole arrangement, including engine, pipes, gas-making apparatus, etc.?

2. Suppose a certain Clerk gas engine requires 26 cubic feet of gas per hour per indicated HP, find its apparent efficiency. If the same engine required 31 cubic feet per hour per brake HP, what is its *practical* efficiency?
3. Find the apparent or indicated efficiency of an Otto gas engine using 22 cubic feet of coal-gas per hour per indicated HP. If the same engine consumed 27·5 cubic feet of gas per hour per brake HP, what is its real efficiency?
4. If an Atkinson gas engine uses 20 and 25 cubic feet of gas per indicated and per brake HP respectively, find the efficiencies as above.
5. Find the consumption of coal per hour per indicated HP in each of the above cases.
6. Supposing an Otto engine uses 82·5 cubic feet of Dowson gas per hour per indicated HP, and that the other engines use proportionate amounts, find the apparent efficiency of each of them with Dowson gas, and the number of lb. of coal burnt by each per hour per indicated HP in that case.

   NOTE.—We are told that 1 cubic foot of Dowson's gas gives out, in complete combustion, 124,000 foot-lb. of mechanical energy, and that 1 lb. of anthracite coal produces 64·27 cubic feet of this gas. It may be taken that 1 lb. of average anthracite, if very carefully burnt, would give out about 11,000,000 foot-lb. of energy.
7. Find the total efficiency of the combination in the case of an Otto engine with Dowson gas generator, using gas equivalent to 1·3 lb. of anthracite per hour per indicated HP.
8. An Otto gas engine has a piston 12 inches in diameter, 8-inch crank, speed 150 revolutions per minute, and the mean pressure in cylinder as found from the indicator diagram (after due allowance has been made for the in-drawing and expression parts of the cycle) is 62·2 lb. per square inch. Find the indicated HP, there being an explosion every two revolutions.

   NOTE.—In working out the indicated HP of a steam engine we took the mean velocity of the piston to be the

number of feet passed through by it per minute—whilst acted on *by pressure from the steam*. Here the mean piston velocity—for this purpose—is also taken as the distance the piston moves, in feet, every minute *whilst acted on by pressure from the burning gas*, in this case $\frac{16}{12} \times 75 =$ 99·76 feet per minute. The mean total pressure on the piston multiplied by 99·76 and divided by 33,000 will, therefore, give the indicated HP, the Otto cycle giving a working stroke every two revolutions.

9. It was found that when the engine was working with this amount of power it used 533 cubic feet of coal-gas per hour. What was the indicated efficiency, and what the coal consumption?
10. In a small Otto gas engine the consumption of coal-gas was 48 cubic feet per hour. The diameter of the piston was 5·75 inches, length of crank 3 inches, speed 160·3 revolutions per minute, mean pressure 54·31 lb. per square inch. If the brake HP was 1·3, find the practical efficiency of the engine, the indicated HP, the fraction of the indicated HP that appears on the dynamometer, and the gas consumption per brake HP per hour.
11. Assuming that the working fluid in the gas engine follows approximately the law $\frac{P\,l}{T} =$ constant, where P is the pressure in lb. per square inch, l the length of the cylinder occupied by the gas, including clearance, and T its absolute temperature; and being given that in an Otto engine, when $P = 14·7$ lb. per square inch (the ordinary atmospheric pressure), $l = 2·222$ feet, and $T = 673°$, find the temperature at the highest part of the diagram, where $p = 92·65$ lb. per square inch, and $l = 1·1$ foot.
12. Taking the temperature found in the last Example as the temperature of the source, and the temperature of the exhaust (about 400° C.) as the temperature of the refrigerator, find the maximum efficiency a heat engine could possibly have, working between these limits of temperature.

NOTE.—The efficiency of a perfect engine working between the absolute temperatures T_1 and T is $\frac{T_1 - T}{T_1}$. Ordinary centigrade temperature + 273 = absolute temperature.

13. If 180 gallons per hour is the quantity of water supplied to the water jacket, and if this water came into the jacket at 15° C., and left at 60° C., find the energy wasted per minute, and also the HP wasted in heating this water.

14. In order to find the law of the expansion curve of a gas-engine indicator diagram, a number of ordinates were drawn at right angles to the atmospheric line, and the pressure p (above vacuum), together with the corresponding length of the cylinder space (including clearance) behind the piston, was observed for the position of the piston represented by each ordinate. Knowing that the law is probably of the form $p\,l^x =$ constant, the logarithms of corresponding values of p and l were plotted as coordinates of points on squared paper. It was found that these points lay fairly well on a straight line. From two points on that line the following values were obtained:—

When log. $p = 1{\cdot}8$, log. $l = 1{\cdot}346$;
and when log. $p = 2{\cdot}3$, log. $l = 1{\cdot}028$.

From these data find the law of the expansion curve of the diagram.*

15. In a Priestman oil engine (Otto cycle), the cylinder being $8\frac{1}{2}$ inches in diameter, piston 12 inches' stroke, speed 212 revolutions per minute, average effective pressure in the cylinder 47·36 lb. per square inch, oil consumption 6·5 pints (say pounds) per hour, the oil having a heat value of about 19,000 British thermal (Fahrenheit) units per lb., find the indicated HP, the efficiency of the engine as between energy supplied and power

* The data introduced in these examples are, by kind permission, taken from Professor William Robinson's "Gas and Petroleum Engines."

indicated, and the cost of power per indicated HP per hour, the oil costing $4\frac{1}{2}$d. per gallon. If the brake HP was 7·05, find the cost per brake HP per hour, and the actual efficiency of the engine.

ANSWERS.

1. Efficiency of Lenoir engine, ·039. Total efficiency, ·01. It must be remembered, however, that the coke and other products obtained in gas-making are valuable.
2. Apparent efficiency, ·143 or 14·3 per cent. Practical or useful efficiency, 12·05 per cent.
3. Indicated efficiency, 16·98 per cent. Real or practical efficiency, 13·59 per cent.
4. Indicated efficiency, 18·67 per cent. Real efficiency, 14·95 per cent.
5. Lenoir, 21·1 lb. of coal per hour per indicated HP.
   Clerk, 5·77 lb. " " "
   Otto, 4·88 lb. " " "
   Atkinson, 4·4 lb. " " "
6. Atkinson { Apparent efficiency, 21·28 per cent.
   1·168 lb. of coal per hour per indicated HP.
   Otto { 19·35 per cent. " " "
   1·285 lb. of coal " " "
   Clerk { 16·29 per cent. " " "
   1·52 lb. of coal " " "
   Lenoir { 4·46 per cent. " " "
   5·56 lb. of coal " " "
7. Apparent efficiency of combination, 13·85 per cent.
8. 21·317 indicated HP.
9. Indicated efficiency, 14·94 per cent. Coal consumption, 5·55 lb. per hour per indicated HP.
10. Real efficiency, 10·12 per cent. Indicated HP, 1·71. 76 per cent. of indicated HP appears on dynamometer. 37 cubic feet per brake HP per hour.
11. $T = 2{,}100°$ absolute temperature or 1827° C.
12. Maximum efficiency, 68 per cent.
13. Energy wasted per minute, 187,050 foot-lb. 56·8 HP thus wasted.
14. Law of expansion curve, $p\ l^{1·572} = 8·248$.
15. 8·63 HP indicated. ·18 efficiency (indicated). Cost, ·42 penny per indicated HP per hour. Cost, ·5 penny per *brake* HP per hour. Actual efficiency, 14·8 per cent.

LESSON XXXVII.

CENTRIFUGAL FORCE. CENTRIFUGAL GOVERNORS.

MATTER, if in motion, tends to move on in a straight line with a uniform velocity, except acted on by force. Hence, if a mass be compelled to move in a circular path, there must be a force or forces acting on it, their resultant being towards the centre of the circle, hence called the "centripetal" force. The force exerted *by the body*

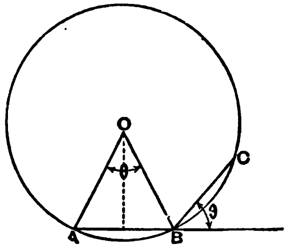


Fig. 77.

itself is equal and opposite to this, and is called the "centrifugal" force.

The following elementary investigation of the change of velocity—really the change in the *direction* of motion—of the body may be interesting. A more rigid, but less easily understood, proof can be given by introducing the notation of the Differential Calculus.

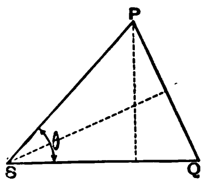


Fig. 78

Instead of moving with a regularly changing direction, let the body move along the sides of a regular polygon, as in Fig. 77; moving along A B with a uniform velocity, then suddenly changing its direction to B C, and so on. Let S Q and S P (Fig. 78) represent the uniform velocity along A B and B C, then P Q will represent the acceleration * or change

* It is doubtful if "acceleration" is the proper term to use here, though it represents something which is the result of force acting on mass.

of velocity at B. But from the figures it is easily seen that the triangles P S Q and A O B are similar, hence $\frac{PQ}{QS} = \frac{AB}{AO}$, or alternately $\frac{PQ}{AB} = \frac{QS}{AO}$.

But since P Q represents the centripetal acceleration a, and A B represents the uniform velocity v—

$$\therefore \frac{PQ}{AB} = \frac{QS}{AO}$$

may be written—

$$\frac{a}{v} = \frac{v}{r}, \text{ or } a = \frac{v^2}{r}.$$

Let the number of sides in the polygon be indefinitely increased, in the limit the body moves with a uniform velocity on the circle, and the centripetal acceleration is found *by dividing the square of the velocity in feet per second by the radius of the circular path in feet.*

$$\text{But force} = \text{mass} \times \text{acceleration.}$$

$$\therefore \text{the centripetal force} = \frac{m\,v^2}{r},$$

the *centrifugal* force exerted *by the body itself* being opposite to this—*i.e.*, directed radially *outwards*—and of the same amount.

If A is the angular velocity in radians per second, $v = A\,r$, and centrifugal force $= m\,A^2\,r$, neglecting sign as regards direction.

If n is the number of revolutions per minute, then—
$A = \frac{2\,\pi\,n}{60}$, and centripetal or centrifugal force $= \frac{W\,r\,n^2}{2{,}936}$.

CENTRIFUGAL GOVERNORS.

Perhaps one of the most useful applications of the principles above referred to is that of the centrifugal governor used on steam and other engines. In its simplest form, as invented by James Watt, it is shown in Fig. 79, and will be readily understood.

The student will easily see that by a simple application of the "triangle of forces," if F is the

centrifugal force of one ball, r the distance of its centre from the axis, and w the weight of the ball—

$$\frac{F}{w} = \frac{r}{h}, \text{ or } F = \frac{w\,r}{h};$$

and since

$$F = \frac{w\,r\,n^2}{2{,}936},$$

$$n = \sqrt{\frac{2{,}936}{h}},$$

friction being neglected.

More usually the governor is loaded, say by a constant

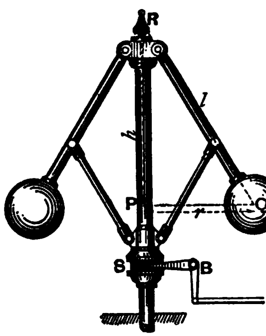


Fig. 79.

load W being placed on the sleeve S, and surrounding the central spindle. This device increases the force available to do the work of moving heavy parts and overcoming friction in the ratio of $\frac{W}{2} + w$ to w,* and also increases the speed of the governor in the ratio of the square roots of these quantities, on the supposition that the vertical movement of the two weights is the same; which would be the case if the links go from the *centres* of the suspending rods to the sleeve and are equal to half the former in length. If, on the other hand, each

* See the article on the Steam Engine in the "Encyclopædia Britannica."

ball is joined by two equal rods to the axis and to the sleeve, the ratio is W + w to w. There are other governors, such as Hartnell's, in which a powerful spring is used instead of the constant load W, the balls being supported on bell-crank levers so that their weight is practically eliminated, but their centrifugal force (or mass) is available as before.

If an experiment be made with a governor, the balls being pulled out radially by two equal forces—these forces and the corresponding values of r being noted, both as the balls are *opening out* and as they are closing—the results, if plotted, will give two curves, with a region between due to friction. The whole cycle of operations for the governor can then be followed on the curves. Slightly different values would, of course, be correct if the governor revolved.

Numerical Examples.

1. Find the centrifugal force of a spherical ball of cast iron, 3 inches in diameter, when its centre moves on a circle of 2 feet radius, the ball making 150 revolutions per minute. A cubic inch of cast iron weighs ·27 lb., and the volume of a sphere is ·5236 times the cube of its diameter.
2. Find the total centrifugal force of the two balls of a Watt governor, the balls being of cast iron, 3 inches in diameter, when rotating with their centres at 8·5 inches from the axis, and at a speed of 90 revolutions per minute. If a constant weight of 20 lb. is placed on the sleeve of the governor, find the centrifugal force and speed necessary to keep the balls rotating at the same radius (friction being neglected), if the rods are arranged respectively as described above.
3. A locomotive passes round a railway curve of 500 feet radius at a speed of 30 miles an hour. If the weight on one pair of wheels is 16 tons, find the outward or centrifugal force of these wheels on the rails, supposing them free to move outwards, and neglecting the fact that centrifugal force acts

through the centre of the moving mass. If the centres of the rails are 5 feet apart, how much will the outer rail have to be elevated so that the resultant of gravity and centrifugal force may act perpendicularly to the plane of the surface of the rails? In such a question as this is it necessary to know the weight of the train?

4. Show that Rankine's rule, by which the elevation of the outer rail should be $= \frac{\text{gauge} \times V^2}{15r}$, where V is the speed in miles per hour, is approximately correct.

5. The following numbers were obtained by experimenting with a certain steam-engine governor:—

Distance of centre of ball from axis $= r$ feet.	Centrifugal force in lb. $=$ F.
0·167	6 $\pm$ 1·05
0·25	9·3 $\pm$ 1·15
0·333	13·08 $\pm$ 1·3
0·417	19·8 $\pm$ 1·45

The weight of the two balls was 4 lb. Find the range of speed which will alter the balls from $r =$ ·292 to $r =$ ·417 and back again. (The student will require to use squared paper as already explained.)

HARTNELL'S GOVERNOR.

6. Push of spring $\pm$ friction $=$ centrifugal force. Let friction $= \pm$ 6 lb. Find the size of cast-iron balls, two of which will have a centrifugal force $\frac{r\,n^2}{360}$.

When r changes from $1\frac{1}{4}$ foot to 1 foot, let a complete change occur in the gear—smallest to fullest steams.

1st. Find size of spring which, with no friction, would make the governor isochronous for 200 revolutions per minute.

Radius of coils, 1·5 inch. Weight producing permanent set, 300 lb. Modulus of rigidity, $N = 13 \times 10^6$. Shear stress of permanent set, 120,000 lb. per square inch.

2nd. If the spring is slackened one inch for the position $r = 1$, find the fluctuation of speed for friction, and compare with total fluctuation.

EXPLANATION OF EXAMPLES 5 AND 6.

No. 5. Plotting the given values of r and F as the co-ordinates of points on squared paper, two curves are obtained. Suppose the values of r are plotted as abscissæ, and the corresponding values of F (two for each value of r) as ordinates, then it will easily be seen that the centrifugal force or speed can increase by an amount represented by the vertical distance between the two curves at any point without any change taking place in the distance of the balls for the centre. The whole cycle of operations can easily be traced on the curves.

No 6. An isochronous governor, without friction, would give a straight line through the origin, instead of the curves noticed above. This means that F is proportional to r or $F = a\,r$. Also from the question we have—

$$F = \frac{r\,n^2}{360} \therefore a = \frac{n^2}{360}.$$

$$\text{When } n = 200, \text{ centrifugal force} = r \times \frac{(200)^2}{360} = r \times 111{\cdot}1$$
$$\text{or } a = 111{\cdot}1.$$

When $r = 1$ foot, centrifugal force $= 111{\cdot}1$ lb.
When $r = 1{\cdot}25$ feet, centrifugal force $= 138{\cdot}8$ lb.

Hence we are required to find a spring whose push to begin with is 111 lb., and when it is shortened 3 inches its push is to be 138·8 lb. That is, a spring which will just bear a load of 300 lb. without being permanently injured, and which will shorten or lengthen 3 inches for a force of 138·8 − 111·1 lb. It is here assumed that when the balls get farther from the centre by any amount, the spring shortens by the *same* amount. The remainder of the calculation is easily enough performed; knowing the laws of the stiffness and strength of spiral springs (given in Lesson XXIX.). In the last part of the Example, suppose the spring slackened till—

$$F_1 = 102 \text{ lb. instead of } 111 \text{ lb.}$$

and till—

$$F_2 = 130 \text{ lb.} \quad ,, \quad 139 \text{ lb.}$$

Then total $F_1 = 102 \pm 6$, and total $F_2 = 130 \pm 6$.

$$\text{Then } \frac{1 \times n_1^2}{360} = 102 - 6 = 96 \therefore n_1 = \sqrt{96 \times 360}$$

$$\frac{1 \times n_2^2}{360} = 102 + 6 = 108 \therefore n_2 = \sqrt{108 \times 360}$$

$$\frac{1 \times n_3^2}{360} = 130 + 6 = 136 \therefore n_3 = \sqrt{136 \times 360}$$

$$\frac{1 \times n_4^2}{360} = 130 - 6 = 124 \therefore n_4 = \sqrt{124 \times 360}.$$

The "sensitiveness" of a governor is determined by the smallness of the fraction found by dividing the *change* of n necessary to alter the gear from no steam to full steam by the average value of n between these limits.

ANSWERS.

1. 58·2 lb.
2. Centrifugal force, (1) 14·9, (2) 54 lb.; (3) 93 lb. Speed, (2) 171; (3) 225 revolutions per minute.
3. Centrifugal force, 4,309·6 lb. Outer rail must be elevated ·6 foot. No, weight of train is not required.
5. The cycle of operations is as follows:—Balls at $r = \cdot 292$; least speed, 156·9 revolutions per minute; speed increases to 175·1 revolutions without altering the position of the balls (owing to friction); a further increase of speed to 193·4 revolutions opens the balls to $r = \cdot 417$; then speed can fall again to 179·7 without any change in r.
6. Diameter of each ball, 2·61 inches. Length of wire of spring, 313·4 inches. Diameter of wire, ·268 inch. Speeds, $n_1 = 185{\cdot}9$, $n_2 = 197{\cdot}18$, $n_3 = 221{\cdot}27$, $n_4 = 211{\cdot}28$. Hence the fluctuation of speed allowed by the governor owing to friction, for the position $r = 1$ foot, is $\frac{n_2 - n_1}{\text{mean } n} = \frac{11{\cdot}28}{191{\cdot}5} = \frac{1}{17}$ nearly. For the position $r = 1{\cdot}25$, the fluctuation of speed is $\frac{10}{216{\cdot}27} = \frac{1}{22}$ nearly. The

fraction expressing the "sensitiveness" of the governor is in this case $\frac{221 \cdot 27 - 185 \cdot 9}{203 \cdot 58} = \frac{221 - 186}{203 \cdot 6}$ say $= \frac{1}{6}$ nearly. The smaller this fraction is, the more sensitive is the governor.

LESSON XXXVIII.

THERMODYNAMICS, ETC.—MISCELLANEOUS EXAMPLES

REGNAULT'S rules for steam :—

Total heat (H) of 1 lb. $= 606 \cdot 5 + \cdot 305\, t$,
Latent heat (L) of 1 lb. $= 606 \cdot 5 - \cdot 695\, t$,
where t is the temperature in centigrade units.

1. Find in ft.-pounds the total heat and also the latent heat of 10 lb. of saturated steam at the temperature of 120° C.
2. Taking the value of the latent heat for 1 lb. as in the last Example, find the volume in cubic feet of 1 lb. of the steam, taking the volume at 0° C. to be $\frac{1}{62 \cdot 36}$ cubic foot.

The rule is—

$$L = (v_2 - v_1) \times T \times \frac{df}{dt}.$$
$$T = t + 273,$$
$$\text{and } \frac{df}{dt} = \frac{19 \cdot 68}{10^8}(T - 233)^4 \text{ approximately.}$$

In this rule L is in *work* units.

3. Find the pressure of this steam from Rankine's formula. Log. $p = 6 \cdot 1007 - \frac{1{,}517 \cdot 45}{T} - \frac{1{,}225 \cdot 15}{T^2}$, p being the pressure in lb. per square inch.
4. Suppose the boiler of a certain engine evaporates 9 lb. of water per lb. of coal supplied, and that 3 lb. of coal are required per indicated HP per hour. Find the indicated efficiency of the engine, including boiler, etc.; the steam being formed at 120° C.

5. The work done by a cubic foot of steam is $144\, r\left(p_1 \frac{1+\log._\epsilon r}{r} - p_3\right) - 144\, x.$* This is a maximum when $r = \dfrac{p_1}{p_3 + \dfrac{d\,x}{d\,r}}.$ Mr. Willans found that $\dfrac{d\,x}{d\,r} = 10$ (about) for non-condensing engines. What is the best ratio of cut-off in a non-condensing engine, with an initial pressure of 94·7 lb. per square inch, and a back pressure of 15·5 lb. per square inch?

CHANGE OF PRESSURE IN AIR DUE TO CHANGE OF DEPTH.

From what has already been done (page 158) it is evident that if potential is due to height, gravity being the only volumetric force—

$$d\,p = -\,w\,d\,h$$

$$\frac{d\,h}{d\,p} = \frac{-\,2{,}116}{\cdot 0807} \times \frac{1}{p}$$

since $\dfrac{\cdot 0807}{2{,}116} \times p = w$ (the weight of 1 cubic foot of air).

$$\therefore\ h = \frac{-\,2{,}116}{\cdot 0807}\,(\log.\,p + \text{constant}).$$

When $h = o$, let $p = 2{,}116$, and the constant is $= -\log.\,2{,}116$.

$$\therefore\ h = \frac{-\,2{,}116}{\cdot 0807}\,\log.\left(\frac{p}{2{,}116}\right);\ p \text{ is pressure in lb. per square foot.}$$

$$\text{Let } P = \frac{p}{2{,}116},\ h = \frac{-\,2{,}116}{\cdot 0807}\,(\log.\,P),$$

$$= -\,26{,}200 \log.\,P.$$

$$(1)\ \therefore\ P = \epsilon^{\frac{-h}{26{,}200}}.$$

The similar rule for water is approximately—

$$(2)\ p = 43\cdot 2 \times 10^6\left(\epsilon^{\frac{H}{692{,}300}} - 1\right),$$

which becomes—

$$\frac{p}{43\cdot 2 \times 10^6} = \frac{H}{692{,}300},$$

$$\text{or } \frac{p}{62\cdot 4} = H,$$

if H is small.

* For meaning of p_1, etc., see Examples on Steam Engine.

6. Find the pressure in lb. per square inch at a height of 3 miles in the atmosphere.
7. Find at what depth in water the pressure will be 3 atmospheres; find also the depth at which it will be 3 tons per square inch.
8. At what depth in water will the water have the same density that iron usually has?

ADIABATIC FLOW OF AIR.

Taking our law—

$$h + \frac{v^2}{2g} + \int \frac{dp}{w} = \text{constant},$$

and $p\,w^{-\gamma} = \text{constant}$, our adiabatic law,

$$w = \left(\frac{w_o}{p_o^{\frac{1}{\gamma}}}\right) \times p^{\frac{1}{\gamma}}, \text{ call } \left(\frac{w_o}{p_o^{\frac{1}{\gamma}}}\right) \text{ by letter } k.$$

$$\int \frac{dp}{w} = \int \frac{dp \cdot p^{-\frac{1}{\gamma}}}{k} = \frac{p^{-\frac{1}{\gamma}+1}}{k\left(-\frac{1}{\gamma}+1\right)} = \frac{p^{1-\frac{1}{\gamma}}\, p_o^{\frac{1}{\gamma}}}{w_o\left(1-\frac{1}{\gamma}\right)}$$

$$\text{hence } h + \frac{v^2}{2g} + \frac{p_o^{\frac{1}{\gamma}}}{w_o\left(1-\frac{1}{\gamma}\right)}\, p^{1-\frac{1}{\gamma}} = \text{constant}.$$

Where pressure is p_o, and velocity is o gives us the constant, and neglecting h—

$$\frac{v^2}{2g} + \frac{p_o^{\frac{1}{\gamma}}\, p^{1-\frac{1}{\gamma}}}{w_o\left(1-\frac{1}{\gamma}\right)} = \frac{p_o}{w_o\left(1-\frac{1}{\gamma}\right)},$$

$$(3) \text{ or } v^2 = \frac{2g}{w_o\left(1-\frac{1}{\gamma}\right)} \left\{ p_o - p_o^{\frac{1}{\gamma}}\, p^{1-\frac{1}{\gamma}} \right\}.$$

NOTE.—v is the velocity with which the gas flows (adiabatically) from a place where the pressure is p_o to where it is p. w_o is the weight of one cubic foot at the pressure p_o. For air, $\gamma = 1{\cdot}41$; $w_o = {\cdot}0807$, if p_o is atmospheric pressure.

9. Find the circumferential velocity of a fan necessary to produce a pressure of 50 lb. per square inch above the atmosphere. Find also the speed of the fan in revolutions per minute, if it is 10 feet in diameter.

N.B.—The velocity is that with which the air would flow, adiabatically, from where the pressure is 25 lb. per square inch above atmospheric into the atmosphere.

10. In the above case find the velocity and speed necessary to produce a pressure equal to 6 inches of water.

11. Air flows adiabatically from a place where the pressure is 2 atmospheres to where it is $\frac{1}{10}$ of an atmosphere, through a round orifice 1 inch in diameter with sharp edges. If the vena contracta is the same as for water (0·6), find (1) the velocity of the air, (2) the flow in cubic feet per second, (3) the weight of air flowing into the latter space per second.

ANSWERS.

1. Total heat = 9,003,400 ft.-lb. Latent heat = 7,323,400 ft.-lb.
2. Volume = 14·47 cubic feet.
3. 27·97 lb. per square inch. 4. 8·2 per cent.
5. r = 3·8, or the steam is cut off at a little more than $\frac{1}{4}$ of the stroke.
6. 8·03 lb. per square inch.
7. 101·9 feet. 15,336 feet. 8. About 270 miles.
9. v = 1,746 feet per second, or 3,334 revolutions per minute. This shows the impossibility of producing high pressures by fans.
10. 106·3 feet per second, or 203 revolutions per minute.
11. Velocity, 2,032·5 feet per second. Flow, 6·6 cubic feet per second. Weight flowing, ·1036 lb. per second.

APPENDIX.

USEFUL RULES AND CONSTANTS.

RATIO of CIRCUMFERENCE of circle to DIAMETER:— $\pi = 3{\cdot}1416$ nearly. Log. $\pi = {\cdot}49715$.

Area of circle $= \pi r^2 = {\cdot}7854\ d^2$. Log. $\cdot 7854 = \bar{1}{\cdot}89509$.

Area of ellipse $= {\cdot}7854 \times$ product of major and minor axes.

Area of a trapezium $= \frac{1}{2}$ sum of parallel sides $\times$ perpendicular distance between them.

Area of surface of sphere $= 4\pi r^2 = \pi d^2$. Log. $4\pi = 1{\cdot}09921$.

Volume of sphere $= \frac{4}{3}\pi r^3 = {\cdot}5236\ d^3$. Log. $\frac{4}{3}\pi = {\cdot}62209$. Log. $\cdot 5236 = \bar{1}{\cdot}7190$.

Volume of cone or pyramid $= \frac{1}{3}$ area of base $\times$ perpendicular height.

Volume of frustrum of cone or pyramid =

$$\tfrac{1}{3} h\,(B + \sqrt{B\,b} + b);$$

where h is the height of the frustrum, B and b the areas of its ends.

Volume of cylinder or prism = area of right section $\times$ length.

Feet in 1 English mile, 5,280. Log. $5{,}280 = 3{\cdot}72263$.

Feet in 1 nautical mile, 6,080. Log. $6{,}080 = 3{\cdot}78390$.

Pounds in 1 ton, 2,240. Log. $2{,}240 = 3{\cdot}35025$.

The mechanical unit of power is 1 HP, which means 33,000 ft.-lb. of work done in one minute. Log. $33{,}000 = 4{\cdot}51851$.

The electric unit of power is one watt, which means a current of one *ampere* working through a difference of potential of one *volt*. There are 746 watts to one HP.

Mean radius of the earth, about 21×10^6 feet (20,900,000 feet). If its mean density is 5·67, it contains about $13{\cdot}5 \times 10^{24}$ lb.

Value of g at the Equator, 32·088.
Value of g at the Poles, 32·253.
Value of g at London, 32·182—generally taken as 32·2.
Base of Napierian logarithms, $\epsilon = 2\cdot71828 \ldots$ Log. $\epsilon = \cdot4343$.
To convert common into Napierian logarithms, multiply by 2·30258. Log. 2·3026 = ·36222.
One radian = 57·2958 degs. Degrees × ·0175 = radians.
One gallon of water weighs 10 lb. and contains ·16 cubic foot, or 277 cubic inches.

WEIGHTS OF MATERIALS.

Material.	Weight of one cubic foot in lb.	Weight of one cubic inch in lb.
Pure water, at 39° F.	62·425*	·036
River water (average)	63	—
Sea water	64	—
Wrought iron	490	·283
Cast iron	468	·271
Cast steel	496	·287
Rolled brass...	526	·305
Cast brass	518	·299
Copper	555	·321
Gun metal	546	·316
Zinc	449	·26
Phosphor bronze (rolled)	553	·32
Manganese bronze	524	·303
Brickwork	112	·065
Rubble masonry (about)	140	·081
Concrete (average)	150	·087

MOMENTS OF INERTIA OF SOLIDS.

The moment of inertia, I, of a rod of length l about an axis at one end perpendicular to its length is $M\frac{l^2}{3}$, where M is the mass of the rod.

If the axis were through the centre of gravity of the rod, $I = \frac{M\,l^2}{12}$.

* 62·4 is usually taken in examples.

I of a sphere of radius R about an axis through its centre $= \frac{2}{5}$ M R^2.

I of a spherical shell, outside radius R, inside radius r, about a diameter, is $\frac{2}{5} \frac{M(R^5 - r^5)}{(R^3 - r^3)}$.

I of an ellipsoid (major and minor axes, $2a$ and $2b$ respectively), about major axis, is $\frac{2}{5}$ M b^2.

I of a circular cylinder of radius R, about its geometrical axis, is $\frac{M R^2}{2}$.

I of a hollow circular cylinder, outside radius R, inside radius r, about its geometrical axis, is M $\frac{(R^2 + r^2)}{2}$.

I of an elliptical cylinder (semi-axes of ellipse, a and b respectively), about geometrical axis, is M $\frac{a^2 + b^2}{4}$.*

RULES FOR APPROXIMATE CALCULATION.

CONTRACTED MULTIPLICATION OF DECIMALS.

Example.—Find 954·32865 × 23·21141 correct to 3 places of decimals.

Rule.—Omit decimal points, reverse the multiplier, placing the units figure of the new multiplier under the last figure to be retained in the multiplicand (in this case the figure 6); commence by multiplying the right-hand figure of the multiplier into that immediately above it, bringing in any figure which would be carried from the next right-hand figure of the multiplicand.

Proceed in this way with every figure of the multiplier which stands under a figure of the multiplicand, commencing to write each result under the first figure of

* For other moments of inertia, see page 103; or Rankine's "Rules and Tables."

the multiplier. Add up and put in the decimal point to show 3 places of decimals.

```
   9543286|5
   1411232
  ---------
  19086573
   2862985
    190865
      9543
       954
       381
         9
  ---------
  22151·310
```

The position of the decimal point may be verified by what has been called the "*grade*" rule.* Let the *grade* of a number be represented by a numeral *one less* than the number of integer figures contained in the number. Then, in multiplication, if the leading figures in the quotient *rise* in value as compared with the multiplicand, the grade of the product is the *sum of the grades of the factors.* If the leading figures *fall* in value, add one *extra* to the sum of the grades. (The greater of the two factors is taken as the multiplicand.)

For division the rule is: Take the difference of the grades of the dividend and divisor—subtracting one extra when the leading digits of the divisor exceed those of the dividend—to get the grade of the quotient.

CONTRACTED DIVISION OF DECIMALS.

Rule.—Move the decimal point so that the divisor shall be a whole number, making the corresponding change in the dividend. Arrange the dividend, by adding ciphers or rejecting figures, to have the same number of decimals as are required in the answer. Cut off from the right hand as many figures as there are in the divisor *less one.* The part remaining is the dividend proper. Divide till all the figures of the dividend are exhausted, then begin to *contract* by rejecting one figure

* Hensley's "Scholar's Arithmetic."

of the divisor before each multiplication, commencing with the figure at the right-hand side of the divisor. If the divisor is greater than the dividend proper, begin to contract at once. The position of the decimal point in the quotient can be determined by the rule already given.

Example.—Find 246·83412 ÷ 22·8246 to 3 places of decimals (say 4).

```
228246)246834|1·2000 (108144
 .....  228246|
        ------
         18588            Grade of dividend 2
         18259              „       divisor 1
         -----                              -
           329                   difference 1
           228
           ---         Leading digits of divisor do not
           101           exceed those of the dividend,
            91           hence answer is 10·8144.
            --
            10
             9
            --
```

A dot is placed under each figure of the divisor as it is rejected, or a line may be drawn through each figure instead. Also in multiplying the remaining part of the divisor add the number which would have been carried over from the multiplication of the last rejected figure.

The student will observe that the grade of a number, consisting wholly of decimals, is *negative*, and is greater by one than the number of ciphers following the decimal point.

DIFFERENCE OF TWO SQUARES.

The product of the sum and difference of two numbers is equal to the difference of their squares. This rule is especially convenient when the numbers differ only by a small amount. Thus—

$$151^2 - 149^2 = 300 \times 2 = 600.$$

The student should also remember that the square root of such a number as 1·004 is 1·002 very nearly.

Also the error in taking the square of such a number as 1·03 to be 1·06 is not great. The real answer is 1·0609.

FOUR-FIGURE LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	·0000	·0043	·0086	·0128	·0170	·0212	·0253	·0294	·0334	·0374	4	8	12	17	21	25	29	33	37
11	·0414	·0453	·0492	·0531	·0569	·0607	·0645	·0682	·0719	·0755	4	8	11	15	19	23	26	30	34
12	·0792	·0828	·0864	·0899	·0934	·0969	·1004	·1038	·1072	·1106	3	7	10	14	17	21	24	28	31
13	·1139	·1173	·1206	·1239	·1271	·1303	·1335	·1367	·1399	·1430	3	6	10	13	16	19	23	26	29
14	·1461	·1492	·1523	·1553	·1584	·1614	·1644	·1673	·1703	·1732	3	6	9	12	15	18	21	24	27
15	·1761	·1790	·1818	·1847	·1875	·1903	·1931	·1959	·1987	·2014	3	6	8	11	14	17	20	22	25
16	·2041	·2068	·2095	·2122	·2148	·2175	·2201	·2227	·2253	·2279	3	5	8	11	13	16	18	21	24
17	·2304	·2330	·2355	·2380	·2405	·2430	·2455	·2480	·2504	·2529	2	5	7	10	12	15	17	20	22
18	·2553	·2577	·2601	·2625	·2648	·2672	·2695	·2718	·2742	·2765	2	5	7	9	12	14	16	19	21
19	·2788	·2810	·2833	·2856	·2878	·2900	·2923	·2945	·2967	·2989	2	4	7	9	11	13	16	18	20
20	·3010	·3032	·3054	·3075	·3096	·3118	·3139	·3160	·3181	·3201	2	4	6	8	11	13	15	17	19
21	·3222	·3243	·3263	·3284	·3304	·3324	·3345	·3365	·3385	·3404	2	4	6	8	10	12	14	16	18
22	·3424	·3444	·3464	·3483	·3502	·3522	·3541	·3560	·3579	·3598	2	4	6	8	10	12	14	15	17
23	·3617	·3636	·3655	·3674	·3692	·3711	·3729	·3747	·3766	·3784	2	4	6	7	9	11	13	15	17
24	·3802	·3820	·3838	·3856	·3874	·3892	·3909	·3927	·3945	·3962	2	4	5	7	9	11	12	14	16
25	·3979	·3997	·4014	·4031	·4048	·4065	·4082	·4099	·4116	·4133	2	3	5	7	9	10	12	14	15
26	·4150	·4166	·4183	·4200	·4216	·4232	·4249	·4265	·4281	·4298	2	3	5	7	8	10	11	13	15
27	·4314	·4330	·4346	·4362	·4378	·4393	·4409	·4425	·4440	·4456	2	3	5	6	8	9	11	13	14
28	·4472	·4487	·4502	·4518	·4533	·4548	·4564	·4579	·4594	·4609	2	3	5	6	8	9	11	12	14
29	·4624	·4639	·4654	·4669	·4683	·4698	·4713	·4728	·4742	·4757	1	3	4	6	7	9	10	12	13
30	·4771	·4786	·4800	·4814	·4829	·4843	·4857	·4871	·4886	·4900	1	3	4	6	7	9	10	11	13
31	·4914	·4928	·4942	·4955	·4969	·4983	·4997	·5011	·5024	·5038	1	3	4	6	7	8	10	11	12
32	·5051	·5065	·5079	·5092	·5105	·5119	·5132	·5145	·5159	·5172	1	3	4	5	7	8	9	11	12
33	·5185	·5198	·5211	·5224	·5237	·5250	·5263	·5276	·5289	·5302	1	3	4	5	6	8	9	10	12
34	·5315	·5328	·5340	·5353	·5366	·5378	·5391	·5403	·5416	·5428	1	3	4	5	6	8	9	10	11
35	·5441	·5453	·5465	·5478	·5490	·5502	·5514	·5527	·5539	·5551	1	2	4	5	6	7	9	10	11
36	·5563	·5575	·5587	·5599	·5611	·5623	·5635	·5647	·5658	·5670	1	2	4	5	6	7	8	10	11
37	·5682	·5694	·5705	·5717	·5729	·5740	·5752	·5763	·5775	·5786	1	2	3	5	6	7	8	9	10
38	·5798	·5809	·5821	·5832	·5843	·5855	·5866	·5877	·5888	·5899	1	2	3	5	6	7	8	9	10
39	·5911	·5922	·5933	·5944	·5955	·5966	·5977	·5988	·5999	·6010	1	2	3	4	5	7	8	9	10
40	·6021	·6031	·6042	·6053	·6064	·6075	·6085	·6096	·6107	·6117	1	2	3	4	5	6	8	9	10
41	·6128	·6138	·6149	·6160	·6170	·6180	·6191	·6201	·6212	·6222	1	2	3	4	5	6	7	8	9
42	·6232	·6243	·6253	·6263	·6274	·6284	·6294	·6304	·6314	·6325	1	2	3	4	5	6	7	8	9
43	·6335	·6345	·6355	·6365	·6375	·6385	·6395	·6405	·6415	·6425	1	2	3	4	5	6	7	8	9
44	·6435	·6444	·6454	·6464	·6474	·6484	·6493	·6503	·6513	·6522	1	2	3	4	5	6	7	8	9
45	·6532	·6542	·6551	·6561	·6571	·6580	·6590	·6599	·6609	·6618	1	2	3	4	5	6	7	8	9
46	·6628	·6637	·6646	·6656	·6665	·6675	·6684	·6693	·6702	·6712	1	2	3	4	5	6	7	7	8
47	·6721	·6730	·6739	·6749	·6758	·6767	·6776	·6785	·6794	·6803	1	2	3	4	5	5	6	7	8
48	·6812	·6821	·6830	·6839	·6848	·6857	·6866	·6875	·6884	·6893	1	2	3	4	4	5	6	7	8
49	·6902	·6911	·6920	·6928	·6937	·6946	·6955	·6964	·6972	·6981	1	2	3	4	4	5	6	7	8
50	·6990	·6998	·7007	·7016	·7024	·7033	·7042	·7050	·7059	·7067	1	2	3	3	4	5	6	7	8
51	·7076	·7084	·7093	·7101	·7110	·7118	·7126	·7135	·7143	·7152	1	2	3	3	4	5	6	7	8
52	·7160	·7168	·7177	·7185	·7193	·7202	·7210	·7218	·7226	·7235	1	2	2	3	4	5	6	7	7
53	·7243	·7251	·7259	·7267	·7275	·7284	·7292	·7300	·7308	·7316	1	2	2	3	4	5	6	6	7

FOUR-FIGURE LOGARITHMS (*continued*).

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	
54	·7324	·7332	·7340	·7348	·7356	·7364	·7372	·7380	·7388	·7396	1	2	2	3	4	5	6	6	7
55	·7404	·7412	·7419	·7427	·7435	·7443	·7451	·7459	·7466	·7474	1	2	2	3	4	5	5	6	7
56	·7482	·7490	·7497	·7505	·7513	·7520	·7528	·7536	·7543	·7551	1	2	2	3	4	5	5	6	7
57	·7559	·7566	·7574	·7582	·7589	·7597	7604	·7612	·7619	·7627	1	2	2	3	4	5	5	6	7
58	·7634	·7642	·7649	·7657	·7664	·7672	·7679	·7686	·7694	·7701	1	1	2	3	4	4	5	6	7
59	·7709	·7716	·7723	·7731	·7738	·7745	·7752	·7760	·7767	·7774	1	1	2	3	4	4	5	6	7
60	·7782	·7789	·7796	·7803	·7810	·7818	·7825	·7832	·7839	·7846	1	1	2	3	4	4	5	6	6
61	.7853	·7860	·7868	·7875	·7882	·7889	7896	·7903	·7910	·7917	1	1	2	3	4	4	5	6	6
62	·7924	·7931	·7938	·7945	·7952	7959	·7966	·7973	·7980	·7987	1	1	2	3	3	4	5	6	6
63	.7993	·8000	·8007	·8014	·8021	·8028	·8035	·8041	·8048	·8055	1	1	2	3	3	4	5	5	6
64	·8062	·8069	·8075	·8082	·8089	·8096	·8102	·8109	·8116	8122	1	1	2	3	3	4	5	5	6
65	·8129	·8136	·8142	·8149	·8156	·8162	·8169	·8176	·8182	·8189	1	1	2	3	3	4	5	5	6
66	·8195	·8202	·8209	·8215	·8222	·8228	·8235	·8241	·8248	·8254	1	1	2	3	3	4	5	5	6
67	·8261	·8267	·8274	·8280	·8287	·8293	·8299	·8306	·8312	·8319	1	1	2	3	3	4	5	5	6
68	·8325	·8331	·8338	·8344	·8351	·8357	·8363	·8370	·8376	·8382	1	1	2	3	3	4	4	5	6
69	·8388	·8395	·8401	·8407	·8414	·8420	8426	·8432	·8439	·8445	1	1	2	2	3	4	4	5	6
70	·8451	·8457	·8463	·8470	·8476	·8482	·8488	·8494	·8500	·8506	1	1	2	2	3	4	4	5	6
71	·8513	·8519	·8525	·8531	·8537	·8543	·8549	·8555	·8561	·8567	1	1	2	2	3	4	4	5	5
72	·8573	·8579	·8585	·8591	·8597	·8603	·8609	·8615	·8621	·8627	1	1	2	2	3	4	4	5	5
73	·8633	·8639	·8645	·8651	·8657	·8663	·8669	·8675	·8681	·8686	1	1	2	2	3	4	4	5	5
74	·8692	·8698	·8704	·8710	·8716	·8722	·8727	·8733	·8739	·8745	1	1	2	2	3	4	4	5	5
75	·8751	·8756	·8762	·8768	·8774	·8779	·8785	·8791	·8797	·8802	1	1	2	2	3	3	4	5	5
76	·8808	·8814	·8820	·8825	·8831	·8837	·8842	·8848	·8854	·8859	1	1	2	2	3	3	4	5	5
77	·8865	·8871	·8876	·8882	·8887	·8893	·8899	·8904	·8910	·8915	1	1	2	2	3	3	4	4	5
78	·8921	·8927	·8932	·8938	·8943	·8949	·8954	·8960	·8965	·8971	1	1	2	2	3	3	4	4	5
79	·8976	·8982	·8987	·8993	·8998	·9004	·9009	·9015	·9020	·9025	1	1	2	2	3	3	4	4	5
80	·9031	·9036	·9042	·9047	·9053	·9058	·9063	·9069	·9074	·9079	1	1	2	2	3	3	4	4	5
81	·9085	·9090	·9096	·9101	·9106	·9112	·9117	·9122	·9128	·9133	1	1	2	2	3	3	4	4	5
82	·9138	·9143	·9149	·9154	·9159	·9165	·9170	·9175	·9180	·9186	1	1	2	2	3	3	4	4	5
83	·9191	·9196	·9201	·9206	·9212	·9217	·9222	·9227	·9232	9238	1	1	2	2	3	3	4	4	5
84	·9243	·9248	·9253	·9258	·9263	·9269	·9274	·9279	·9284	·9289	1	1	2	2	3	3	4	4	5
85	·9294	·9299	·9304	·9309	·9315	·9320	·9325	·9330	·9335	·9340	1	1	2	2	3	3	4	4	5
86	·9345	·9350	·9355	·9360	·9365	·9370	·9375	·9380	·9385	·9390	1	1	2	2	3	3	4	4	5
87	·9395	·9400	·9405	·9410	·9415	·9420	·9425	·9430	·9435	·9440	0	1	1	2	2	3	3	4	4
88	·9445	·9450	·9455	·9460	·9465	·9469	·9474	·9479	·9484	·9489	0	1	1	2	2	3	3	4	4
89	·9494	·9499	·9504	·9509	·9513	·9518	·9523	·9528	·9533	·9538	0	1	1	2	2	3	3	4	4
90	·9542	·9547	·9552	·9557	·9562	·9566	·9571	·9576	·9581	·9586	0	1	1	2	2	3	3	4	4
91	·9590	·9595	·9600	·9605	·9609	·9614	·9619	·9624	9628	·9633	0	1	1	2	2	3	3	4	4
92	·9638	·9643	·9647	·9652	·9657	·9661	·9666	·9671	·9675	·9680	0	1	1	2	2	3	3	4	4
93	·9685	·9689	·9694	·9699	·9703	9708	·9713	·9717	·9722	·9727	0	1	1	2	2	6	3	4	4
94	·9731	·9736	·9741	·9745	·9750	·9754	·9759	·9763	·9768	·9773	0	1	1	2	2	3	3	4	4
95	·9777	·9782	·9786	·9791	·9795	·9800	9805	·9809	·9814	·9818	0	1	1	2	2	3	3	4	4
96	·9823	·9827	·9832	·9836	·9841	·9845	·9850	·9854	·9859	·9863	0	1	1	2	2	3	3	4	4
97	·9868	·9872	·9877	·9881	·9886	·9890	·9894	·9899	·9903	·9908	0	1	1	2	2	3	3	4	4
98	·9912	·9917	·9921	·9926	·9930	·9934	·9939	·9943	·9948	·9952	0	1	1	2	2	3	3	4	4
99	·9956	·9961	·9965	·9969	·9974	·9978	·9983	·9987	·9991	·9996	0	1	1	2	2	3	3	3	4

TABLE OF ANTILOGARITHMS.

	THIRD FIGURE.										FOURTH FIGURE.								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Table of Antilogarithms (*continued*).

	Third Figure.										Fourth Figure.								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
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